MATHEMATICAL MODELLING OF FLUID-MECHANICS, HEAT-TRANSFER AND CHEMICAL-REACTION PROCESSES: A LECTURE COURSE

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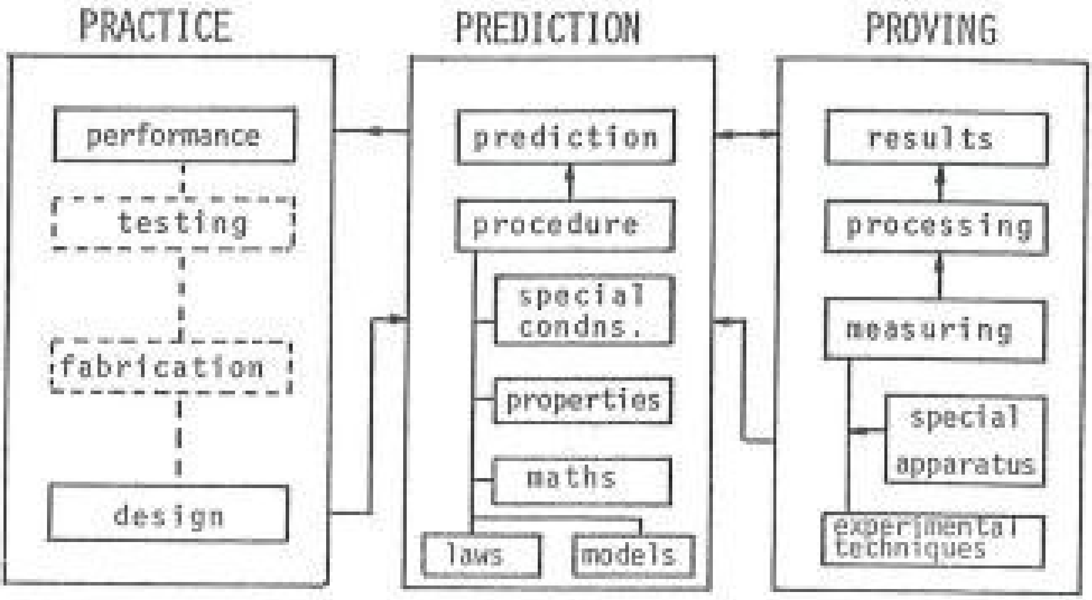
HTE 1 1	2/15	OBJECTIVES OF THE COURSE

- TO PROMOTE UNDERSTANDING OF PROCESSES OF TRANSFER OF CONCENTRATION, HEAT AND MOMENTUM, IN:
  - ENGINEERING EQUIPMENT,
  - THE NATURAL ENVIRONMENT.
- TO FACILITATE QUANTITATIVE PREDICTION OF THESE PROCESSES.
- HENCE TO PERMIT DESIGNS TO BE OPTIMISED, QUICKLY AND ECONOMICALLY.

HTE 1 1	3/15	RELATION OF PREDICTION TO DESIGN & RESEARCH



HTE 1 1	4/15	PROCESSES TO BE PREDICTED - THE NATURAL ENVIRONMENT

- ATMOSPHERE: • SPREAD OF SMOKE FROM CHIMNEY;
  - DISPERSION OF MOIST AIR FROM COOLING-TOWER;
  - WIND FORCES ON BUILDINGS.
- HYDROSPHERE: • THERMAL POLLUTION OF RIVERS;
  - HEATING AND COOLING OF LAKES;
  - DISPERSION OF SEWAGE IN THE OCEAN.
- INSIDE BUILDINGS: • HEATING AND VENTILATING;
  - MOVEMENT OF SMOKE RESULTING FROM FIRES.
- BENEATH THE GROUND: • OIL RECOVERY;
  - SOLUTION MINING.

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HTE 1 .1	5/15	PROCESSES TO BE PREDICTED - AEROSPACE

- TURBO-JET ENGINES: • FLOWS IN COMPRESSORS, TURBINES.
  - COOLING OF TURBINE BLADES.
  - MIXING IN EXHAUST DUCTS.
  - DESIGN OF COMBUSTION CHAMBERS FOR LOW POLLUTION.

- ROCKET ENGINES: • RADIATION FROM EXHAUST PLUMES.
  - UNSTEADY COMBUSTION OF SOLID-PROPELLANT MOTORS.

- AIRCRAFT: • COOLING OF FUSELAGE IN SUPERSONIC FLIGHT.
  - PREDICTION OF AERODYNAMIC FORCES.
  - BOUNDARY LAYERS AT WING-FUSELAGE JUNCTION.
  - AIRFRAME-ENGINE INTEGRATION AND INTERACTION.

HTE 1 .1	6/15	PROCESSES TO BE PREDICTED - SHIP HYDRODYNAMICS

- WAVE DRAG: • INFLUENCES OF HULL SHAPE, SPEED.
  - INFLUENCES OF PITCHING AND ROLLING.

- FRICTION: • INFLUENCES OF HULL SHAPE, SPEED.
  - INFLUENCES OF ROUGHNESS, BIOLOGICAL GROWTH.

- STERN REGION: • PREDICTION OF VELOCITY DISTRIBUTION NEAR STERN AND IN WAKE.
  - DESIGN OF PROPELLERS FOR MAXIMUM EFFICIENCY, LOW NOISE.
  - DESIGN OF RUDDERS.

HTE 1 1	7/15	PROCESSES TO BE PREDICTED - COMBUSTION PROCESSES

- FURNACES: • HEAT-FLUX DISTRIBUTION AT WALLS.
  - EFFICIENCY OF BURNING OF FUELS.
  - INFLUENCE OF FLOW PATTERN ON POLLUTANT FORMATION.

- INTERNAL-COMBUSTION ENGINES: • SPARK-IGNITION.
  - DESIGN OF STRATIFIED-CHARGE GASOLINE ENGINES.
  - INFLUENCE OF CHAMBER DESIGN ON DIESEL COMBUSTION.
  - UNSTEADY FLOW IN AIR-INTAKE DUCTS.

- FIRES: • IGNITION AND SPREAD OF DOMESTIC FIRES.
  - IGNITION AND SPREAD OF FOREST FIRES.
  - EXTINCTION OF FIRES BY CHEMICAL MEANS.

HTE 1 1	8/15	PROCESSES TO BE PREDICTED - INDUSTRIAL EQUIPMENT

- HEAT EXCHANGERS: • STEAM CONDENSERS FOR POWER PLANT.
  - COOLING OF WATER BY CONTACT WITH AIR.
  - PRE-HEATING FUEL OIL BEFORE BURNING.
  - PRE-HEATING AIR BY INDIRECT CONTACT WITH EXHAUST.

- MASS-TRANSFER EQUIPMENT: • DISTILLATION COLUMNS.
  - LIQUID-LIQUID CONTACTERS.
  - TEXTILE-DRYING EQUIPMENT.

- METALLURGICAL PROCESSES: • IRON-ORE REDUCTION.
  - CONTINUOUS CASTING OF ALUMINIUM.

HTE 1 1	9/15	PROCESSES TO BE PREDICTED - DOMESTIC

- HEATING OF BUILDINGS: • HEAT LOSS BY CONDUCTION.
  - USE OF SOLAR COLLECTORS.
  - HEAT PUMPS AND THEIR SOURCES.
  - DISTRICT-HEATING SCHEMES.
- AIR MOVEMENT: • PROMOTION OF DRAUGHT-FREE VENTILATION BY JETS.
  - CIRCULATION OF DEHUMIDIFIED AIR.
- EQUIPMENT: • COOKERS.
  - REFRIGERATORS.
  - AIR PURIFICATION.

HTE 1 1	10/15	PROCESSES TO BE PREDICTED - BIOLOGICAL

- WITHIN THE BODY: • OXYGENATION OF BLOOD.
  - LOCAL SHEAR INTENSITY AT BLOOD-VESSEL WALLS.
  - TEMPERATURE CONTROL BY CAPILLARY CONTRACTION.
  - COOLING BY PERSPIRATION.
- EQUIPMENT: • ARTIFICIAL LUNGS.
  - EQUIPMENT FOR EFFECTING LOCAL FREEZING.
  - COOLING OF DENTISTS' HIGH-SPEED DRILLS.
  - PROVISION OF CONTROLLED ENVIRONMENTS.

HTE 1 1	11/15	PROCESSES TO BE PREDICTED - NUCLEAR POWER

- GAS-COOLED REACTORS: • OPTIMUM COOLING-FIN DESIGN FOR FUEL RODS ENCASED IN METAL.
  - FLOW AND HEAT TRANSFER IN ROD BUNDLES.
- BOILING-WATER REACTORS: • RESPONSE OF SYSTEM TO SUDDEN CHANGES (SHUT-DOWN; LIQUID ESCAPE).
  - TWO-PHASE FLOW IN COMPLEX CIRCUMSTANCES.
- SODIUM-COOLED REACTORS: • INFLUENCE OF A BLOCKAGE ON FLOW AND HEAT TRANSFER IN REACTOR-ROD BUNDLE.
  - CIRCULATION OF SODIUM BY FREE CONVECTION IN VESSEL SURROUNDING REACTOR.

HTE 1 1	12/15	REQUIREMENTS OF PREDICTION PROCEDURES

- TRUTH, I.E. AGREEMENT WITH EXPERIENCE.
- SPEED, WITH ALLOWANCE FOR: LEARNING TIME, SET-UP TIME, EXECUTION TIME, INTERPRETATION.
- ECONOMY, WITH CONSIDERATION OF ALL COSTS.
- ACCESSIBILITY.
- FLEXIBILITY.

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KINDS OF PREDICTION PROCEDURE

- TEST OF IDENTICAL EQUIPMENT AND SITUATION.
- TEST OF COMPONENTS + NETWORK ANALYSIS.
- TEST OF SCALE MODEL + DIMENSIONAL ANALYSIS.
- TEST OF SIMPLIFIED SCALE MODEL + HOPEFUL APPROXIMATIONS.
- SOLUTION OF DIFFERENTIAL EQUATIONS EMBODYING MATHEMATICAL MODELS OF COMPLEX PHENOMENA (UNIVERSAL?).
- SOLUTION OF FUNDAMENTAL DIFFERENTIAL EQUATIONS.

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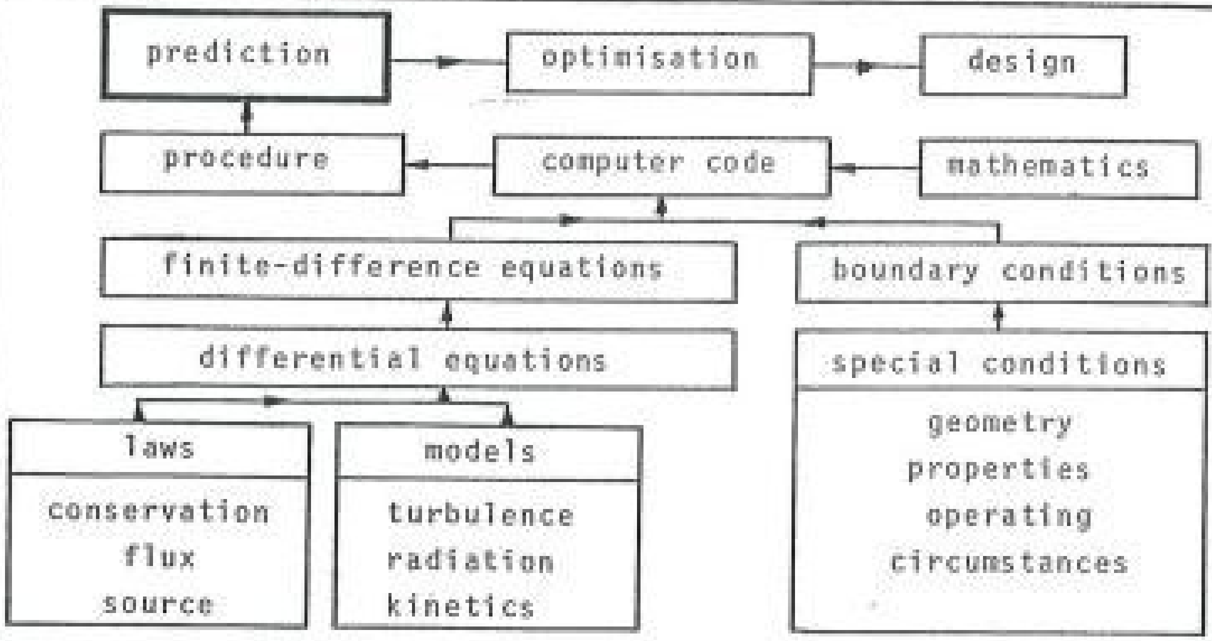
COMPONENTS OF COMPUTER-BASED PREDICTIONS



HTE 1 1	15 ― 15	CURRENT STATUS OF PREDICTIVE ABILITY

- TRUTH: GOOD ENOUGH FOR MANY PURPOSES; BUT CONTINUED RESEARCH ON MODELS IS NEEDED.
- SPEED: EXECUTION TIME SATISFACTORY. LEARNING TIME, ETC., MUST BE REDUCED.
- ECONOMY: EXECUTION COSTS OFTEN TOO HEAVY; REDUCTION EASILY POSSIBLE.
- ACCESSIBILITY: GENERALLY POOR; MECHANISMS ARE BEING CREATED.
- FLEXIBILITY: GOOD, AND IMPROVING.

HTE 1 2	1/15	LECTURE 2. CONSERVATION LAWS. INTRODUCTION.

CONTENTS:
- MASS CONSERVATION.
- MOMENTUM "CONSERVATION".
- CHEMICAL-SPECIES "CONSERVATION".
- ENERGY "CONSERVATION".

NOTES :
- "CONSERVATION" ≡ BALANCE OF FACTORS EFFECTING CHANGES.
- THESE LAWS ARE ENTIRELY RELIABLE (MORE SO THAN TRANSPORT LAWS, ETC.).
- VECTOR, AND CARTESIAN-TENSOR, FORMS WILL BE PROVIDED.

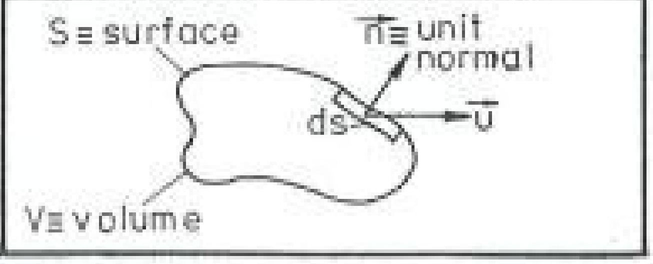
HTE 1 2	2/15	MASS CONSERVATION (OFTEN CALLED "CONTINUITY").

- IN WORDS: RATE OF INCREASE OF DENSITY = <u>NET</u> RATE OF INFLOW OF MASS INTO UNIT VOLUME.
- IN VECTOR NOTATION: $\boxed{\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \vec{u}) = 0}$
- NOMENCLATURE: ρ ≡ DENSITY (MASS/VOLUME).

  $\vec{u}$ ≡ VELOCITY VECTOR ≡ MASS CROSSING UNIT AREA IN UNIT ÷ DENSITY.

  t ≡ TIME.

  $$\mathrm{div}\,\vec{u} \equiv \lim_{V\to 0} \frac{1}{V} \int_S \vec{n}.\vec{u}\, dS$$



HTE 1 2	3/15	MASS CONSERVATION; CARTESIAN-TENSOR FORM.

- CONTINUITY EQUATION: $\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.}$
- NOTATION: • x_i STANDS FOR x_1, x_2, x_3 (x,y,z).
- u_i STANDS FOR u_1, u_2, u_3 (u,v,w).
- i STANDS FOR 1, 2, 3.
- IF INDEX APPEARS TWICE, SUM OF 3 TERMS IS IMPLIED.
- REPEATED INDICES ARE CALLED "DUMMY SUBSCRIPTS".
- TERMS WITHOUT DUMMY SUBSCRIPTS ARE SCALARS.
- TERMS WITH ONE DUMMY SUBSCRIPT ARE VECTORS.
- ILLUSTRATION: $\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_1}(\rho u_1) + \frac{\partial}{\partial x_2}(\rho u_2) + \frac{\partial}{\partial x_3}(\rho u_3) = 0.$

HTE 1 2	4/15	MASS CONSERVATION; "BULK-CONTINUITY" FORM.

- REARRANGEMENT PRODUCES: $\boxed{\mathrm{div}\,\vec{u} = -\frac{D}{Dt}(\ell n\rho)}$
- NOTATION: • $\frac{Ds}{Dt} \equiv \frac{\partial s}{\partial t} + \vec{u}.\mathrm{grad}\,s \equiv \frac{\partial s}{\partial t} + u_i\frac{\partial s}{\partial x_i}$.
- $\frac{D}{Dt}$ ≡ "SUBSTANTIAL DERIVATIVE" ≡ RATE OF CHANGE FOR A PARTICULAR "PARCEL" OF MATTER.
- IMPORTANCE: • FOR MANY FLOWS, RHS = 0 EVEN THOUGH ρ VARIES WITH t AND x_i, E.G. 2 LIQUIDS.
- OFTEN $D(\ell n\rho)/Dt$ IS EASIER TO CALCULATE THAN ITS COMPONENTS.

N.B. s ≡ ANY SCALAR.

HTE 1 2	5 — 15	MOMENTUM "CONSERVATION"; (NEWTON'S SECOND LAW OF MOTION).

- IN WORDS, FOR THE m'TH DIRECTION: RATE OF INCREASE OF m - DIRECTION MOMENTUM (MDM) PER UNIT VOLUME = NET RATE OF INFLOW OF MDM PLUS NET FORCE, BOTH PER UNIT VOLUME.

- IN VECTOR NOTATION:

$$\frac{\partial}{\partial t}(\rho u_m) + \mathrm{div}\,(\rho \vec{u}\, u_m) = \mathrm{div}\,\vec{p}_m + g_m - f_m$$

- NOMENCLATURE: • u_m ≡ MDM PER UNIT MASS,
  - p_m ≡ FLUID-STRESS (PRESSURE, SHEAR) VECTOR,
  - g_m ≡ BODY FORCE IN DIRECTION m PER UNIT VOLUME,
  - f_m ≡ RESISTIVE FORCE IN DIRECTION m PER UNIT VOLUME.

HTE 1 2	6 — 15	MOMENTUM "CONSERVATION"; CARTESIAN-TENSOR FORM.

- DIFFERENTIAL EQUATION:

$$\frac{\partial}{\partial t}(\rho u_m) + \frac{\partial}{\partial x_i}(\rho u_i u_m) = \frac{\partial}{\partial x_i} p_{m,i} + g_m - f_m \,.$$

- NOTES: • $p_{m,i}$ IS THE i-DIRECTION COMPONENT OF ALL THE FLUID STRESSES AFFECTING m-DIRECTION MOMENTUM.
  - OFTEN, BUT NOT NECESSARILY, m WILL BE ONE OF THE i'S.
  - BODY FORCE g_m INCLUDES EFFECTS OF GRAVITY, ELECTRO-MAGNETIC FORCES.
  - RESISTANCE FORCE f_m IS FINITE WHEN THE FLUID IS PENETRATING A POROUS MEDIUM.
  - IN LATTER CASE $u_m > u_i$ EVEN WHEN m = i.

HTE 1 / 2	7/15	MOMENTUM "CONSERVATION"; COMBINATION OF MOMENTUM AND MASS EQUATIONS.

- VECTOR FORM: $\frac{\partial u_m}{\partial t} + \vec{u}.\text{grad } u_m = \frac{1}{\rho} (\text{div } \vec{p}_m + g_m - f_m)$
- ALTERNATIVE: $\frac{Du_m}{Dt} = \frac{1}{\rho} (\text{div } \vec{p}_m + g_m - f_m)$
- GENERAL REMARK: • THE SAME PHYSICAL FACTS CAN BE EXPRESSED IN MANY ALTERNATIVE MATHEMATICAL FORMS.
  - WHICH FORM IS CONVENIENT DEPENDS UPON WHAT PROBLEM IS TO BE SOLVED, AND BY WHAT METHOD.
  - ONLY FEW FORMS ARE PROVIDED IN THESE LECTURES.

HTE 1 / 2	8/15	CHEMICAL-SPECIES "CONSERVATION"

- IN WORDS: RATE OF INCREASE OF MASS OF CHEMICAL SPECIES ℓ = NET RATE OF CONVECTIVE + DIFFUSIVE INFLOW + NET RATE OF CHEMICAL PRODUCTION, ALL PER UNIT VOLUME.
- VECTOR FORM: $\boxed{\frac{\partial}{\partial t} (\rho m_\ell) + \text{div } (m_\ell \rho \vec{u} + \vec{J}_\ell) = R_\ell}$
- NOMENCLATURE: • $m_\ell \equiv$ MASS FRACTION OF ℓ,
  - $\vec{J}_\ell \equiv$ DIFFUSION FLUX OF $\ell \equiv$ TOTAL FLUX $- m_\ell \rho \vec{u}$ (kg/m²s),
  - $R_\ell \equiv$ CHEMICAL PRODUCTION RATE OF ℓ (kg/m³s),
- NOTES: • $\sum_{\text{all } \ell} m_\ell = 1$, • $\sum_{\text{all } \ell} \vec{J}_\ell = 0$, • $\sum_{\text{all } \ell} R_\ell = 0$.

HTE 1 2	9/15	CHEMICAL-SPECIES "CONSERVATION"; ALTERNATIVE FORMS.

- CARTESIAN TENSOR FORM: $\boxed{\frac{\partial}{\partial t}\rho m_\ell + \frac{\partial}{\partial x_i}(\rho u_i m_\ell + J_{\ell,i}) = R_\ell}$
- COMBINATION WITH CONTINUITY:

$$\rho\left(\frac{\partial m_\ell}{\partial t} + \vec{u}.\text{grad } m_\ell\right) \equiv \rho\frac{Dm_\ell}{Dt} = -\text{div } \vec{J}_\ell + R_\ell$$

- ALTERNATIVE VARIABLES:
  - A SYMBOL CAN BE ASCRIBED TO ρm_ℓ (E.G. ρ_ℓ), WITH OBVIOUS CONSEQUENCES.
  - COMPOSITION CAN BE DESCRIBED IN TERMS OF MOLE FRACTION, PARTIAL PRESSURE; BUT DIFFICULTIES ARISE.

HTE 1 2	10/15	CHEMICAL-SPECIES "CONSERVATION"; RELATION TO CHEMICAL-ELEMENT CONSERVATION.

- NOMENCLATURE: • m_α ≡ MASS OF ELEMENT α/MASS OF MIXTURE. $\equiv \sum_{\text{all } \ell} m_\ell \; m_{\alpha,\ell}$, WHERE, $m_{\alpha,\ell}$ ≡ MASS OF ELEMENT α/MASS OF SPECIES ℓ.
- NOTE: m_α VARIES; $m_{\alpha,\ell}$ IS A CONSTANT FOR SPECIES ℓ.
- INSERTION IN EQUATION OF PANEL 8, WITH SUMMATION FOR ALL SPECIES IN THE MIXTURE AFTER MULTIPLICATION BY $m_{\alpha,\ell}$.

$$\boxed{\frac{\partial}{\partial t}(\rho m_\alpha) + \text{div}\left(m_\alpha \rho \vec{u} + \sum_{\text{all } \ell} m_{\alpha,\ell}\,\vec{J}_\ell\right) = 0}$$

- NOTE: • $\sum_{\text{all } \ell} m_{\alpha,\ell} R_\ell$ HAS BEEN PUT = 0 BECAUSE CHEMICAL REACTION CANNOT CREATE OR DESTROY ELEMENTS.

HTE 1 / 2	11 / 15	ENERGY "CONSERVATION"; (FIRST LAW OF THERMODYNAMICS)

- IN WORDS: RATE OF INCREASE OF (INTERNAL ENERGY + KINETIC ENERGY) = NET RATE OF INFLOW OF STAGNATION ENTHALPY + NET RATE OF (HEAT + SHEAR WORK) + (RADIATION + ELECTRICAL + OTHER SOURCES), ALL PER UNIT VOLUME.

- VECTOR FORM:

$$\frac{\partial}{\partial t}\{\rho(\tilde{h} - \frac{p}{\rho})\} + \text{div}\,(\rho\vec{u}\tilde{h} + \vec{q} + \vec{W}_s + \sum_{\text{all } \ell} h_\ell \vec{J}_\ell) = S_{rad} + \ldots$$

- NOMENCLATURE: • $\tilde{h} \equiv h + \vec{u}.\vec{u}/2$
  - $h \equiv \sum_{\text{all } \ell} m_\ell h_\ell$, • $h_\ell \equiv h_{\ell,o} + \int_{T_o}^{T} c_\ell(T)dT$
  - $\vec{q}$ ≡ heat flux vector, $\vec{W}_s$ ≡ shear-work vector, S ≡ source.

HTE 1 / 2	12 / 15	ENERGY "CONSERVATION"; ALTERNATIVE FORMS.

- CARTESIAN TENSORS:

$$\frac{\partial}{\partial t}(\rho\tilde{h} - p) + \frac{\partial}{\partial x_i}(\rho u_i\tilde{h} + Q_i + W_{s,i} + \sum_{\text{all } \ell} h_\ell J_{\ell,i}) = S_{rad} + \ldots$$

- COMBINATION WITH CONTINUITY:

$$\rho\frac{D\tilde{h}}{Dt} + \frac{\partial}{\partial x_i}(Q_i + W_{s,i} + \sum_{\text{all } \ell} h_\ell J_{\ell,i}) = -\frac{\partial p}{\partial t} + S_{rad} + \ldots$$

- SINCE $dh \equiv \sum_\ell (m_\ell\, c_\ell\, dT + h_\ell\, dm_\ell)$, THE FIRST TERM CAN BE REWRITTEN.

$$\rho\left\{(\sum_\ell m_\ell\, c_\ell)\frac{DT}{Dt} + \sum_\ell h_\ell \frac{Dm_\ell}{Dt} + \frac{D}{Dt}(\vec{u}.\vec{u}/2)\right\}$$

HTE 1 2	13/15	ENERGY "CONSERVATION"; NOTES.

- AT LOW MACH NOS. (MORE STRICTLY, ECKERT NOS.) w_s CAN BE NEGLECTED; AND $\partial p/\partial t$ MAY BE SMALL.
- SINCE $\sum_{all\ \ell} m_\ell c_\ell \equiv c\ (T)$, THE MIXTURE SPECIFIC HEAT AT CONSTANT PRESSURE, $dh \equiv c dT + \sum_{all\ \ell} (h_{\ell,o} + \int_{T_o}^{T} c\ dT)\ dm_\ell$.
- IF ALL c_ℓ'S ARE EQUAL AT THE SAME TEMPERATURE,
  $dh \equiv c dT + \sum_{all\ \ell} h_{\ell,o}\ dm_\ell$,
  BECAUSE $\sum_{all\ \ell} dm_\ell = 0$.

HTE 1 2	14/15	GENERAL FORM OF CONSERVATION EQUATION

- NOMENCLATURE: LET ϕ STAND FOR u_m, m_ℓ, $\tilde{h}$, OR UNITY (FOR CONTINUITY EQUATION).
- GENERAL EQUATION: ALL THE FOREGOING EQUATIONS CAN BE WRITTEN AS:

$$\boxed{\frac{\partial}{\partial t}(\rho\phi) + \text{div}\,(\rho\vec{u}\,\phi + \vec{J}_\phi) = s_\phi}$$

- NOTES: • FOR $\phi \equiv 1$, $\vec{J}_\phi = 0$ AND $s_\phi = 0$.
  - FOR $\phi \equiv m_\ell$, $s_\phi = R_\ell$.
  - FOR $\phi \equiv \tilde{h}$, $\vec{J}_\phi = \vec{Q} + \vec{w}_s + \ldots$ AND $s_\phi = s_{rad} + \ldots$
  - FOR $\phi \equiv u_m$, $\vec{J}_\phi$ = part of div $\vec{p}_m$ AND s_ϕ = remaining part + $g_m - f_m$.

HTE 1 2	$\frac{15}{15}$	CONCLUDING REMARKS ABOUT CONSERVATION EQUATIONS

- ALL THE EQUATIONS EXHIBIT SIMILARITIES OF FORM.
- OTHER EQUATIONS OF THE SAME FORM ARE ENCOUNTERED ELSEWHERE IN PHYSICS, E.G.:
  - LAW OF CONSERVATION OF ELECTRICAL CHARGE;
  - LAW OF "CONSERVATION" OF TURBULENCE ENERGY.
- BEFORE THE EQUATIONS CAN BE USED, EXPRESSIONS MUST BE FOUND PERMITTING THE EVALUATION OF J_ϕ (TRANSPORT LAWS) AND OF S_ϕ (SOURCE LAWS).

HTE 1 / 3	1/15	LECTURE 3. FLUX LAWS. INTRODUCTION.

CONTENTS:
- HEAT CONDUCTION.
- DIFFUSION OF MATTER.
- VISCOUS ACTION.
- RADIATIVE TRANSFER.

NOTES: • THE "FLUX LAWS" PROVIDE THE EXPRESSIONS FOR $\vec{J}_{g}$, $\vec{q}$, $\vec{P}_m$, ETC.

• THE LAWS FOR ALL FOUR TRANSFER PROCESSES EXHIBIT SIMILARITIES, WHICH WILL BE EMPHASISED AND USED; BUT THEY ARE FAR FROM BEING IDENTICAL.

HTE 1 / 3	2/15	FOURIER'S LAW OF HEAT CONDUCTION

- IN WORDS: HEAT IS TRANSFERRED OPPOSITELY TO THE TEMPERATURE GRADIENT, IN PROPORTION TO THAT GRADIENT; THE PROPORTIONALITY CONSTANT DEPENDS ON THE LOCAL PROPERTIES OF THE MEDIUM.
- IN SYMBOLS: $\boxed{\vec{q} = -\lambda \text{ grad } T}$
- NOMENCLATURE: • $\lambda \equiv$ THERMAL CONDUCTIVITY (J/ms°C).
  - LATER, Γ_h WILL BE INTRODUCED, DEFINED BY:

$$\Gamma_h \equiv \lambda / c \qquad \text{(kg/ms)}$$

  - Γ_h IS CALLED THE EXCHANGE COEFFICIENT FOR HEAT.

HTE 1 3	3/15	FOURIER'S LAW; REFINEMENTS.

- EFFECTS OF OTHER GRADIENTS: IN GASES, HEAT MAY BE TRANSFERRED AS A CONSEQUENCE OF GRADIENTS OF CONCENTRATION AND PRESSURE. THE EFFECTS ARE SMALL.

- TURBULENCE: IN TURBULENT FLOW, EDDY EXCHANGE TRANSFERS ENERGY IN A MANNER SIMILAR TO HEAT CONDUCTION; THEN WE WRITE $\boxed{\vec{q} = -\lambda_{eff}\ \text{grad}\ T}$, TIME-AVERAGE QUANTITIES BEING UNDERSTOOD.

- RELATION TO OTHER TRANSPORT PROPERTIES:
  - $c\mu/\lambda$ ≡ PRANDTL NO. DIMENSIONLESS; μ ≡ VISCOSITY.
  - $c\mathcal{D}\rho/\lambda$ ≡ LEWIS NO., DIMENSIONLESS; $\mathcal{D}$ ≡ DIFFUSION COEFFICIENT.

HTE 1 3	4/15	HEAT CONDUCTION; SOURCES OF DATA.

- FOR GASES, $c\mu/\lambda \approx .7$; SO λ IS OBTAINED FROM VISCOSITY DATA.

- FOR ALL FLUIDS, $.5 \leq c\mu_{eff}/\lambda_{eff} \leq 1$; SO λ_{eff} IS OBTAINED FROM EFFECTIVE-VISCOSITY DATA.

- FOR ALL MATERIALS, COMPILATIONS EXIST IN TEXTBOOKS AND HAND-BOOKS, E.G., REID R C, SHERWOOD T K, "PROPERTIES OF GASES AND LIQUIDS", McGRAW HILL, 1958; PERRY J H (ED.), "CHEMICAL HANDBOOK", McGRAW HILL, 1963.

- NOTE: FURTHER INFORMATION ABOUT λ_{eff}, THE EFFECTIVE CONDUCTIVITY IN TURBULENT FLOWS, FOLLOWS IN LECTURE 8.

HTE 1 3	5/15	FICK'S LAW OF DIFFUSION

- IN WORDS: DEFINED SPECIES IS TRANSFERRED BY DIFFUSION OPPOSITELY TO THE MASS-FRACTION GRADIENT, IN PROPORTION TO THAT GRADIENT; THE PROPORTIONALITY CONSTANT DEPENDS ON LOCAL PROPERTIES OF SPECIES AND MEDIUM.
- IN SYMBOLS: $\boxed{\vec{J}_\ell = -\Gamma_\ell \text{ grad } m_\ell}$
- NOMENCLATURE: • $\Gamma_\ell \equiv$ EXCHANGE COEFFICIENT (kg/ms).
  - MORE USUALLY, $\mathcal{D}_\ell$ IS USED (m^2/s), WHERE $\mathcal{D}_\ell = \Gamma_\ell/\rho$.
  - $\mathcal{D}_\ell$ IS CALLED THE DIFFUSION COEFFICIENT OF ℓ.

HTE 1 3	6/15	FICK'S LAW; REFINEMENTS.

- EFFECTS OF OTHER GRADIENTS: ESPECIALLY IN GASES, SPECIES ℓ MAY DIFFUSE AS A CONSEQUENCE OF GRADIENTS OF OTHER SPECIES, OF TEMPERATURE (THERMAL DIFFUSION) AND OF PRESSURE. THE EFFECTS ARE SOMETIMES IMPORTANT.
- TURBULENCE: IN TURBULENT FLOW, EDDY EXCHANGE TRANSFERS SPECIES IN A MANNER SIMILAR TO DIFFUSION; THEN WE WRITE: $\vec{J}_\ell = -\Gamma_{eff,\ell} \text{ grad } m_\ell$, TIME-AVERAGE QUANTITIES BEING UNDERSTOOD.
- RELATION TO OTHER TRANSPORT PROPERTIES: • $\mu/\Gamma_\ell \equiv$ SCHMIDT NO. OF SPECIES ℓ (DIMENSIONLESS).
  - $\Gamma_\ell/\Gamma_h \equiv$ LEWIS NO. OF SPECIES ℓ (DIMENSIONLESS).

HTE 1 3	$\frac{7}{15}$	FICK'S LAW; SOURCES OF DATA.

- SOURCES ARE SIMILAR TO THOSE FOR HEAT CONDUCTION, BUT:

  $\Gamma_\ell < \Gamma_h$ FOR LIQUIDS;

  $\Gamma_\ell << \Gamma_h$ FOR SOLIDS.

- Γ_ℓ IS INDEPENDENT OF PRESSURE FOR GASES (WHEREAS $\mathcal{D}_\ell \propto 1/p$).

- FOR TURBULENT FLUIDS (LIQUID OR GAS),

$$\Gamma_{\ell,eff} \approx \Gamma_{h,eff}$$

  UNDER ALL CONDITIONS.

HTE 1 3	$\frac{8}{15}$	NEWTON'S LAW OF VISCOUS ACTION

- IN WORDS: STRESSES DEVELOP IN A DEFORMING LIQUID IN OPPOSITION TO THE DEFORMATION AND PROPORTIONATE TO THE RATE OF DEFORMATION; THE PROPORTIONALITY CONSTANT DEPENDS ON LOCAL FLUID PROPERTIES.

- IN SYMBOLS: $\vec{\tau}_1 = \mu\Big[\text{grad } u_1 + \vec{i}_1 (\vec{i}_1 . \text{grad } u_1) + \vec{i}_2 (\vec{i}_1 . \text{grad } u_2) + \vec{i}_3 (\vec{i}_1 . \text{grad } u_3) - \frac{2}{3} \vec{i}_1 \text{ div } \vec{u}\Big]$

- NOMENCLATURE: • $\vec{\tau}_1 \equiv \vec{p}_1 + \vec{i}_1 p$, THE VISCOUS-STRESS VECTOR,
  - $\vec{i}_i \equiv$ UNIT VECTOR IN DIRECTION i,
  - $p \equiv$ HYDROSTATIC PRESSURE (N/m^2),
  - $\mu \equiv$ VISCOSITY OF FLUID (kg/ms).

HTE 1 3	9/15	NEWTON'S VISCOSITY LAW; DISCUSSION.

- THE FIRST TERM OF $\tau_1 = \mu \text{ grad } u_1 + \ldots$ IS REMINISCENT OF FOURIER'S AND FICK'S LAWS, WITH $\vec{\tau}_1$ ≡ NEGATIVE OF THE MOMENTUM FLUX.
- IN SIMPLE FLOWS (E.G. BOUNDARY LAYERS) THE FIRST TERM IS OFTEN THE ONLY IMPORTANT ONE.
- $\vec{\tau}_1$ CONTAINS ALL THE PARTS OF $\vec{p}_1$ ASSOCIATED WITH μ.
- IN CARTESIAN COORDINATES, THE COMPONENTS OF $\vec{\tau}_1$ ARE:-

$$\tau_{1,1} = \mu \left(2 \frac{\partial u_1}{\partial x_1} - \frac{2}{3} \text{div } \vec{u} \right),$$

$$\tau_{1,2} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right),$$

$$\tau_{1,3} = \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right).$$

TERMS IMPORTANT IN BOUNDARY LAYERS

HTE 1 3	10/15	NEWTON'S VISCOSITY LAW; SOURCES OF DATA.

- FOR ALL FLUIDS, SEE REID AND SHERWOOD, PERRY, ETC.
- FOR GASES, $\mu \approx \frac{1}{3} \rho \, u_{mol} \, \ell_{mol}$, WHERE u_{mol} ≡ RMS MOLECULAR VELOCITY AND ℓ_{mol} ≡ MEAN FREE PATH.
- FOR TURBULENT FLUIDS,

  (1) EFFECTIVE SHEAR STRESSES ARE PRESUMED TO OBEY NEWTON'S LAW, AND DO SO APPROXIMATELY;

  (2) $\mu_{eff} \propto \rho \, k^{\frac{1}{2}} \ell$, WHERE
  k ≡ KINETIC ENERGY OF TURBULENCE,
  ℓ ≡ LENGTHSCALE OF TURBULENCE.

HTE 1 3	11 15	RADIATIVE TRANSFER; DEFINITIONS.

- RADIANT-ENERGY FLUX, $J_{\lambda,\phi}$: THIS IS THE ENERGY TRANSMITTED PER UNIT AREA ($J/m^2 s$), PER UNIT INCREMENT OF WAVELENGTH λ, AND PER UNIT INCREMENT OF SOLID ANGLE ϕ.
- EMISSIVE POWER, E_λ: THIS EQUALS A FUNCTION OF WAVE LENGTH λ AND TEMPERATURE, TIMES T^4, WHERE $T \equiv$ ABSOLUTE TEMPERATURE.
- ABSORPTIVITY, $a(m^{-1})$, AND SCATTERING COEFFICIENT $s(m^{-1})$: THESE EQUAL THE PROPORTION OF INCIDENT RADIATION ABSORPTION/SCATTERED PER UNIT BEAM LENGTH.

HTE 1 3	12 15	RADIATIVE TRANSFER; LAWS.

- VOLUMETRIC SOURCE OF ENERGY:

$$S_{rad} = \iint a(\lambda)\, J_{\lambda,\phi}\, d\lambda d\phi - \int a(\lambda)\, E_\lambda d\lambda \ ,$$

- NOTES: • SCATTERING DOES NOT ENTER THIS RELATION; FOR IT MERELY CHANGES THE ANGLE OF RADIATION, BUT DOES NOT ALTER ITS INTENSITY.
  - UNDER RESTRICTED CIRCUMSTANCES, EMISSION, REABSORPTION AND SCATTERING OF RADIATION HAVE EFFECTS SIMILAR TO THOSE OF HEAT CONDUCTION.
  - a AND s ARE LIKE RECIPROCAL MEAN FREE PATHS.

HTE 1 3	$\frac{13}{15}$	RADIATIVE TRANSFER; PROPERTIES.

- ABSORPTIVITIES AND SCATTERING COEFFICIENTS ARE NEEDED FOR ALL MATERIALS AND WAVE LENGTHS; THEY ARE INCOMPLETELY AVAILABLE.

- FOR SOLIDS, a AND s ARE SO LARGE THAT IT IS MORE USEFUL TO EMPLOY EXPRESSIONS REPRESENTING INTEGRALS OVER DEPTH INTO THE SURFACE.

- SPECIAL TEXTS SHOULD BE CONSULTED, E.G.:
  HOTTEL H C AND SAROFIM A F
  "RADIATIVE HEAT TRANSFER".
  McGRAW HILL, NEW YORK, 1967.

HTE 1 3	$\frac{14}{15}$	OTHER FLUX LAWS

- OHM'S LAW OF ELECTRICAL CONDUCTION:
  ELECTRICAL FLUX = -RESISTIVITY TIMES VOLTAGE GRADIENT.

- D'ARCY'S LAW OF FLOW IN POROUS MEDIA:
  MASS FLUX = -RESISTANCE COEFFICIENT TIMES PRESSURE GRADIENT.

- NOTES: • ANALOGIES BETWEEN FLUX LAWS CAN BE USED AS AIDS TO UNDERSTANDING.

- SOMETIMES QUANTITATIVE USE CAN BE MADE IN PREDICTIONS, BY WAY OF "ANALOGUE" EXPERIMENTS.

HTE 1 3	15 15	FLUX LAWS; CONCLUDING REMARKS.

- SIMILARITIES BETWEEN THE FLUX LAWS EXIST, AND CAN BE EMPHASISED WITH ADVANTAGE.
- THE NEXT TASK IS TO SUBSTITUTE FLUX LAWS INTO CONSERVATION LAWS TO GET THE DIFFERENTIAL EQUATIONS WHICH MUST BE SOLVED.
- THE FLUX LAWS ARE LESS RELIABLE THAN THE CONSERVATION LAWS (E.G. INFLUENCES ARE OFTEN NEGLECTED).
- CONTINUING RESEARCH WILL REFINE KNOWLEDGE OF THE RELEVANT PROPERTIES.

HTE 1 4	$\frac{1}{15}$	LECTURE 4. SOURCE LAWS. INTRODUCTION.

CONTENTS:
- CHEMICAL REACTION.
- ENERGY SOURCES.
- MOMENTUM SOURCES.
- MASS SOURCES.
- RADIATION SOURCES.
- VOLUME SOURCES.

NOTES:
- THE SOURCE LAWS PROVIDE EXPRESSIONS FOR s_ϕ IN THE GENERAL DIFFERENTIAL EQUATION.
- THESE EXPRESSIONS EXHIBIT GREAT VARIETY.
- DEPARTURES FROM SIMILARITY IN FLUX TERMS CAN BE COMPENSATED IN THE SOURCE TERMS.

HTE 1 4	$\frac{2}{15}$	CHEMICAL-KINETIC LAWS

- GENERAL FORM: $R_\ell = R_\ell\{m_\ell, m_n, \ldots, T, p, \ldots\}$
- EXAMPLE:
  - REACTION: $A + B \rightarrow C$ (IRREVERSIBLE).
  - KIND: CONTROLLED BY FREQUENCY OF SUFFICIENTLY ENERGETIC COLLISION OF A AND B MOLECULES IN GASEOUS FORM.
  - RATE EXPRESSION.

$$R_A = -S\rho^2\, m_A m_B\, N^2 d^2 M_A \left(\frac{\pi \mathcal{R} T}{M^*}\right)^{\frac{1}{2}} \exp\left(\frac{-E}{\mathcal{R}T}\right)$$

HTE 1 4	3/15	CHEMICAL-KINETIC LAWS (CONTINUED)

- DEFINITIONS: • s ≡ STERIC FACTOR, PROPORTION WITH CORRECT ORIENTATION (<1).
  - d ≡ MEAN MOLECULAR DIAMETER ($\approx 3.5 \times 10^{-10}$m).
  - $N \equiv 6.022 \times 10^{26}$ ≡ NO. OF MOLECULES IN 1 KG MOLE.
  - M_A ≡ MOLECULAR WEIGHT OF A.
  - $\mathscr{R}$ ≡ UNIVERSAL GAS CONSTANT, 8314.3J/kg mole°K.
  - M^* ≡ HARMONIC MEAN M, I.E. $1/(1/M_A + 1/M_B)$.
  - E ≡ ACTIVATION ENERGY, OF ORDER 10^8J/kg mole.

- SIMPLIFICATION: OFTEN, THE RATE EXPRESSION IS SIMPLIFIED TO:
  $R_A = -Zm_A m_B\, p^2 \exp\{-E'/(\mathscr{R}T)\}$.

HTE 1 4	4/15	CHEMICAL-KINETIC LAWS; DISCUSSION.

- GENERALISATION: $R_A = -Zm_A{}^a\, m_B{}^b\, p^c \exp\{-E'/(\mathscr{R}T)\}$ WHERE Z, a, b, c AND E' ARE CONSTANTS OF THE REACTION.

- COMPLEXITY OF REACTION SCHEMES: USUALLY MANY SPECIES PARTICIPATE IN SIMULTANEOUS REACTIONS; THE IDEAL REACTION
  fuel + oxygen → product
  NEVER TAKES PLACE.

- THE NEED FOR "MODELS": BECAUSE DATA ARE INCOMPLETE AND TIME LIMITED, OFTEN IDEALISED MODELS OF CHEMICAL REACTION ARE USED.

HTE 1 4	$\frac{5}{15}$	ENERGY SOURCES; "KINETIC HEATING".

- SHEAR-WORK VECTOR:
  $-\vec{W}_s = \vec{\tau}_1 u_1 + \vec{\tau}_2 u_2 + \vec{\tau}_3 u_3$
- COMPONENTS:

$$W_{s,1} = \tau_{1,1}u_1 + \tau_{2,1}u_2 + \tau_{3,1}u_3$$
$$= \mu \left[(2\frac{\partial u_1}{\partial x_1} - \frac{2}{3}\,\mathrm{div}\,\vec{u})u_1 + (\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2})u_2 + (\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3})u_3 \right]$$

$W_{s,2} = \ldots.$

- KINETIC-HEATING SOURCE $= -\mathrm{div}\,\vec{W}_s$.
- IN A SIMPLE CASE, $-\mathrm{div}\,\vec{W}_s = \frac{\partial}{\partial x_2}(\mu\, u_1 \frac{\partial u_1}{\partial x_2})$.

HTE 1 4	$\frac{6}{15}$	OTHER ENERGY SOURCES

- RADIATION: SEE PANEL 12 OF LECTURE 3.
- CHEMICAL REACTION: THIS APPEARS IMPLICITLY, THROUGH THE DIFFUSION FLUXES OF CHEMICAL SPECIES.
- TURBULENCE ENERGY: THIS IS USUALLY NEGLIGIBLE.
- ELECTRICAL DISSIPATION $= (\vec{J}_{elec}.\vec{J}_{elec})/\lambda_{elec}$.
- CONTACT WITH AN INTERSPERSED MEDIUM:
  USUALLY $s_h = \alpha(T_{med} - T)$ WHERE α IS A VOLUMETRIC HEAT-TRANSFER COEFFICIENT.

HTE 1 4	7/15	ENERGY SOURCES; DISCUSSION.

- QUESTION: WHY IS THERE NOT A "CHEMICAL-ENERGY-SOURCE" TERM $\sum h_{\ell,o} R_\ell$ IN THE EQUATION?
- ANSWER: THE HEATS OF FORMATION $h_{\ell,o}$ APPEAR IMPLICITLY IN MANY TERMS IN THE ENERGY EQUATION (PANEL 2.11); AND $\mathrm{div}(\sum_{\mathrm{all}\ j} h_\ell \vec{J}_\ell)$ CAN BE WRITTEN AS:

  $\sum_{\mathrm{all}\ j} (h_\ell \,\mathrm{div}\, \vec{J}_\ell + \vec{J}_\ell . \mathrm{grad}\, h_\ell)$.

  MANIPULATION CAN LEAD TO $\sum h_{\ell,o} R_\ell$, AND OTHER TERMS.

HTE 1 4	8/15	MOMENTUM SOURCES

- PRESSURE GRADIENT: • THIS IS THE SECOND OF THE TWO COMPONENTS OF $\mathrm{div}\ \vec{p}_m$, VIZ. (FROM PANEL 3.8): $\mathrm{div}\ \vec{p}_m = \mathrm{div}(\vec{\tau}_m - \vec{i}_m\, p)$.
- THE MOMENTUM SOURCE FROM PRESSURE GRADIENT IS THUS $-\vec{i}_m . \mathrm{grad} p$, OR $-\frac{\partial p}{\partial x_m}$.
- VISCOUS TERMS: • LATER, THE MAJOR VISCOUS TERMS WILL BE EXPRESSED AS $\mathrm{div}(\mu\ \mathrm{grad}\ u_m)$; THE REMAINDER OF $\mathrm{div}(\vec{\tau}_m)$ WILL BE TREATED AS SOURCE TERMS.

  • THE REASON IS TO STRESS UNIFORMITY.

HTE 1 4	9 15	MOMENTUM SOURCES (CONTINUED)

- INTERNAL RESISTANCE: • DEFINITION: $f_m \equiv F_m u_m$.
  - MEANING: RESISTIVE FORCE PER UNIT VOLUME IS PROPORTIONAL TO THE LOCAL VELOCITY.
  - NON-LINEARITY: IN GENERAL, F_m WILL DEPEND UPON u_m; FOR EXAMPLE, AT HIGH REYNOLDS NO. IN A POROUS MEDIUM, $F_m \alpha |u_m|$.
  - WHEN TWO FLUIDS ARE INTERSPERSED, THE RELATION MAY BE: $f_m \equiv F_m\,(u_m - U_m)$, WHERE U_m IS THE VELOCITY OF THE SECOND FLUID.

HTE 1 4	10 15	MOMENTUM SOURCES (CONTINUED)

- GRAVITATIONAL: • THE SOURCE IS $g_m = \rho\,\vec{i}_m \cdot \vec{a}_{grav}$ WHERE $\vec{a}_{grav} \equiv$ GRAVITATIONAL ACCELERATION VECTOR.
- OFTEN IT IS CONVENIENT TO REDEFINE ρ SO THAT THE GRAVITATIONAL TERM APPEARS AS:

$$(\rho - \bar{\rho})\,\vec{i}_m \cdot \vec{a}_{grav}$$

WHERE $\bar{\rho}$ IS A REFERENCE DENSITY.

- MOVING-COORDINATE TERMS:
- CURVILINEAR COORDINATE TERMS:

} THESE APPEAR IN MORE GENERAL TREATMENTS THAN THE PRESENT ONE.

HTE 1 4	11/15	RADIATION SOURCES

- EMISSION: • THE RADIATION SOURCE REPRESENTED BY $a(\lambda)E_\lambda$ (PANEL 3.12) IS NORMALLY EMITTED UNIFORMLY WITH RESPECT TO DIRECTION.
  - THE INCREMENT INTO THE RADIATION OF ANGLE INTERVAL $\delta\phi$ IS THUS $a(\lambda)E_\lambda\ \delta\phi/(4\pi)$.
- SCATTERING: • DIFFUSE SCATTERING (I.E. EQUAL AT ALL ANGLES) IS RARE.
  - SCATTERING THEREFORE TRANSFERS RADIATION FROM ONE ANGLE INTERVAL TO ANOTHER.

HTE 1 4	12/15	MASS SOURCES

- IN THE STRICT SENSE, MASS SOURCES DO NOT OCCUR; BUT IT IS USEFUL TO ENLARGE THE CONTINUITY CONCEPT TO INCLUDE THEM, FOR TWO REASONS.
- FIRST, WHEN TWO FLUIDS ARE INTERSPERSED, MATERIAL MAY BE TRANSFERRED FROM ONE AND TO THE OTHER.
- EXAMPLE: WATER DROPLETS ARE SUSPENDED IN STEAM; VAPORISATION AND CONDENSATION OCCUR.

HTE 1 4	13/15	MASS SOURCES (CONTINUED)

- SECONDLY, IN NUMERICAL WORK THE CONTINUITY EQUATION IS OFTEN NOT SATISFIED IN THE EARLY PART OF THE COMPUTATION; THE "RESIDUALS" IN THE EQUATIONS ARE CONVENIENTLY THOUGHT OF AS "MASS SOURCES".

- THE CONTINUITY EQUATION CAN THUS BE USEFULLY GENERALISED TO:

$$\frac{\partial}{\partial t}\rho + \mathrm{div}\,(\rho\vec{u}) = S_{mass}$$

WHERE S_{mass} (kg/m^3s) IS NOT ALWAYS ZERO.

HTE 1 4	14/15	VOLUME SOURCES

- FROM PANEL 2.4, THE "BULK-CONTINUITY" EQUATION IS:

$$\mathrm{div}\,\vec{u} = -\frac{D}{Dt}(\ln\rho).$$

- THE RHS IS THE "VOLUME SOURCE PER UNIT VOLUME".

- WHEN A FLUID CHANGES DENSITY ONLY AS A RESULT OF A TEMPERATURE INCREASE ($-\rho^{-1}\,d\rho = \beta dT$), AND THE TEMPERATURE INCREASE ARISES FROM A VOLUMETRIC HEAT SOURCE $\dot{q}'''$, THERE RESULTS THE USEFUL EQUATION:

$$\boxed{\mathrm{div}\,\vec{u} = \beta\,\dot{q}'''/(c\rho)}\;.$$

HTE 1 4	15/15	CONCLUDING REMARKS ABOUT SOURCE LAWS

- THE BALANCE (I.E. "CONSERVATION") EQUATIONS CONTAIN TERMS OF FOUR KINDS:

  - TRANSIENT ($\partial/\partial t$),
  - CONVECTIVE,
  - DIFFUSIVE ,
  - SOURCE .

- NOW THAT ALL FOUR HAVE BEEN TREATED, THE DIFFERENTIAL EQUATIONS CAN BE CONSTRUCTED.

- THE POLICY WILL BE TO EXPRESS THE FIRST THREE TERMS IN SIMILAR FASHIONS, AND TO CARRY THE BURDEN OF VARIETY IN THE FOURTH.

HTE 1 5	1/15	LECTURE 5. THE FUNDAMENTAL DIFFERENTIAL EQUATIONS. INTRODUCTION.

CONTENTS:
- MOMENTUM "CONSERVATION".
- CHEMICAL-SPECIES "CONSERVATION".
- ENERGY "CONSERVATION".
- GENERAL "CONSERVATION" EQUATION.

NOTE:
- THE MASS-CONSERVATION EQUATION HAS ALREADY BEEN SUFFICIENTLY DESCRIBED.

HTE 1 5	2/15	MOMENTUM "CONSERVATION"; VECTOR FORM.

- FROM PANELS 2.5 AND 3.9, WITH $m \equiv i$:

$$\frac{\partial}{\partial t}\rho u_i + \mathrm{div}\,\{\rho\vec{u}\,u_i - \mu\big[\mathrm{grad}\,u_i - \tfrac{2}{3}\vec{i}_i\,\mathrm{div}\,\vec{u} + \vec{i}_i(\vec{i}_i.\mathrm{grad}\,u_i)$$
$$+ \vec{i}_j\,(\vec{i}_i.\mathrm{grad}\,u_j) + \vec{i}_k\,(\vec{i}_i.\mathrm{grad}\,u_k)\big]\} + \vec{i}_i.\mathrm{grad}\,p = g_i - f_i$$

- A SIMPLIFIED FORM IS:

$$\boxed{\frac{\partial}{\partial t}(\rho u_i) + \mathrm{div}\,(\rho\vec{u}\,u_i) = \mathrm{div}\,(\mu\,\mathrm{grad}\,u_i) + S_i}$$

WHERE $S_i \equiv -\vec{i}_i.\mathrm{grad}\,p + g_i - f_i$

$$+ \mathrm{div}\,\{\mu\big[-\tfrac{2}{3}\vec{i}_i\,\mathrm{div}\,\vec{u} + \vec{i}_i\,(\vec{i}_i.\mathrm{grad}\,u_i)$$
$$+ \vec{i}_j\,(\vec{i}_i.\mathrm{grad}\,u_j) + \vec{i}_k\,(\vec{i}_i.\mathrm{grad}\,u_k)\big]\}$$

HTE 1 / 5	3/15	MOMENTUM "CONSERVATION"; DISCUSSION.

- FOR INCOMPRESSIBLE FLOW, $\operatorname{div} \vec{u}$ VANISHES.
- WHEN μ IS INDEPENDENT OF POSITION, THE WHOLE OF THE REMAINDER OF $\operatorname{div} (\mu[\ldots])$ VANISHES.
- THERE ARE OTHER CIRCUMSTANCES UNDER WHICH THIS TERM IS SMALL ENOUGH TO BE IGNORED.
- THE EQUATION IS ESSENTIALLY NON-LINEAR, BECAUSE VELOCITY PRODUCTS APPEAR; ALSO f_i MAY BE NON-LINEAR.
- FOR SOLUTION, SPECIFICATION IS REQUIRED OF ρ, p, μ AS FUNCTIONS OF POSITION AND TIME.

HTE 1 / 5	4/15	MOMENTUM "CONSERVATION"; CARTESIAN TENSOR FORM.

- DIFFERENTIAL EQUATION ($m \equiv i$; GENERAL IS TOO DIFFICULT):

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}\left(\rho u_j u_i + p\delta_{ij} - \mu\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{1}{3} e_{kk}\, \delta_{ij}\right]\right) = g_i - f_i$$

- NOMENCLATURE: • $\delta_{ij} \equiv 1$ IF $i = j$, OTHERWISE 0.

  • $e_{ij} \equiv \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, THEREFORE $e_{kk} = \frac{2\partial u_k}{\partial x_k}$.
- NOTES: • THIS NOTATION IS MORE COMPACT THAN THE VECTOR NOTATION, IN THIS CASE.

  • $\frac{\partial u_k}{\partial x_k} \equiv \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \operatorname{div} \vec{u}$.

HTE 1 5	$\frac{5}{15}$	MOMENTUM "CONSERVATION"; SPECIAL CASES.

- FOR AN INVISCID FLUID ($\mu = 0$):

  $$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) + \frac{\partial p}{\partial x_i} = g_i .$$

- FOR AN INVISCID UNIFORM-ρ FLUID:

  $$\frac{\partial}{\partial t} u_i + \frac{\partial}{\partial x_j}(u_j u_i) = \frac{1}{\rho}\left(-\frac{\partial p}{\partial x_i} + g_i\right).$$

- WHEN CONVECTION TERMS ARE ALSO UNIMPORTANT:

  $$\frac{\partial}{\partial t} u_i = \frac{1}{\rho}\left(-\frac{\partial p}{\partial x_i} + g_i\right).$$

- WHEN ONLY THE INNER RESISTANCE OPPOSES PRESSURE GRADIENTS:

  $$\frac{\partial p}{\partial x_i} = -f_i .$$

HTE 1 5	$\frac{6}{15}$	CHEMICAL-SPECIES "CONSERVATION"; VECTOR FORM.

- FROM PANELS 2.8 AND 3.5:

  $$\boxed{\frac{\partial}{\partial t}(\rho m_\ell) + \mathrm{div}\,(\rho \vec{u}\, m_\ell) = \mathrm{div}\,(\Gamma_\ell\ \mathrm{grad}\ m_\ell) + R_\ell}$$

- THE ALTERNATIVE, FORMED BY COMBINATION WITH CONTINUITY, IS:

  $$\frac{\partial m_\ell}{\partial t} + \vec{u}.\mathrm{grad}\ m_\ell = \frac{1}{\rho}\{\mathrm{div}(\Gamma_\ell\ \mathrm{grad}\ m_\ell) + R_\ell\} \quad .$$

- IN TERMS OF THE SUBSTANTIAL DERIVATIVE:

  $$\frac{D}{Dt} m_\ell = \frac{1}{\rho}\{\mathrm{div}(\Gamma_\ell\ \mathrm{grad}\ m_\ell) + R_\ell\} \quad .$$

HTE 1 5	7/15	CHEMICAL-SPECIES "CONSERVATION"; DISCUSSION.

- SIMILARITY TO MOMENTUM EQUATION: THE m_ℓ EQUATION "NATURALLY" TAKES THE FORM INTO WHICH THE MOMENTUM EQUATION WAS "FORCED".

- NON-LINEARITY: PROBABLY R_ℓ IS NON-LINEAR, BUT NOT CERTAINLY.

- LINKAGES: • $\rho\vec{u}$ LINKS m_ℓ AND u_i EQUATIONS;
  - R_ℓ PROBABLY LINKS MANY MASS-FRACTION EQUATIONS.
  - ρ, Γ_ℓ, ETC. PROBABLY DEPEND ON m's, T, p.

- AUXILIARY INFORMATION: KNOWLEDGE OF ρ, $\vec{u}$, Γ_ℓ, ETC. IS NEEDED FOR THE EQUATION TO BE SOLUBLE.

HTE 1 5	8/15	CHEMICAL-SPECIES "CONSERVATION"; CARTESIAN-TENSOR FORM.

- DIFFERENTIAL EQUATION:

$$\frac{\partial}{\partial t}(\rho m_\ell) + \frac{\partial}{\partial x_i}(\rho u_i m_\ell) = \frac{\partial}{\partial x_i}\left(\Gamma_\ell \frac{\partial m_\ell}{\partial x_i}\right) + R_\ell \quad .$$

- ALTERNATIVE, AFTER COMBINATION WITH CONTINUITY:

$$\frac{\partial m_\ell}{\partial t} + u_i \frac{\partial m_\ell}{\partial x_i} = \frac{1}{\rho}\left\{\frac{\partial}{\partial x_i}\left(\Gamma_\ell \frac{\partial m_\ell}{\partial x_i}\right) + R_\ell\right\} \quad ,$$

- NOTES: • THE EQUATION IS SIMPLE IN <u>ANY</u> FORM.
  - THIS IS WHY IT IS ADOPTED AS THE PATTERN INTO WHICH OTHERS ARE FORCED.

HTE 1 5	9/15	CHEMICAL-SPECIES "CONSERVATION"; SPECIAL CASES.

- FOR A CHEMICALLY-INERT SPECIES, $R_\ell = 0$:

$$\frac{Dm_\ell}{Dt} = \frac{1}{\rho}\{\mathrm{div}\,(\Gamma_\ell\ \mathrm{grad}\ m_\ell)\} \quad ,$$

- IF, ADDITIONALLY, DIFFUSION IS ABSENT:

$$\frac{Dm_\ell}{Dt} = 0 \ .$$

- IF CONVECTION AND REACTION ARE ABSENT:

$$\frac{\partial m_\ell}{\partial t} = \frac{1}{\rho}\ \mathrm{div}\ (\Gamma_\ell\ \mathrm{grad}\ m_\ell) \quad .$$

- IF Γ_ℓ IS UNIFORM: $\frac{\partial m_\ell}{\partial t} = \frac{\Gamma_\ell}{\rho}\ \mathrm{div\ grad}\ m_\ell \quad ,$

HTE 1 5	10/15	CHEMICAL-ELEMENT CONSERVATION; VECTOR FORM.

- FROM PANELS 2.10 AND 3.5:

$$\frac{\partial}{\partial t}\ (\rho m_\alpha) + \mathrm{div}\ (\rho \vec{u}\ m_\alpha) = \mathrm{div} \sum_{\mathrm{all}\ \ell} m_{\alpha,\ell}\ \Gamma_\ell \mathrm{grad}\ m_\ell ,$$

- SIMPLER FORM: IF Γ_ℓ HAS THE SAME VALUE (SAY Γ_α) FOR ALL SPECIES CONTAINING α,

$$\frac{\partial}{\partial t}\ (\rho m_\alpha) + \mathrm{div}\ (\rho \vec{u}\ m_\alpha) = \mathrm{div}\ (\Gamma_\alpha\ \mathrm{grad}\ m_\alpha).$$

- NOTE: • NOW m_α OBEYS THE SAME EQUATION AS DOES A CHEMICALLY-INERT SPECIES.

  - SINCE, IN TURBULENT FLOWS, Γ_{eff}'S ARE EQUAL, THIS SIMPLIFICATION IS OFTEN USEFUL.

HTE 1 / 5	11 / 15	ENERGY "CONSERVATION"; VECTOR FORM.

- FROM PANELS 2.12 AND 3.2:

$$\frac{\partial}{\partial t}(\rho\tilde{h}-p) + \text{div}\,(\rho\vec{u}\,\tilde{h}) = \text{div}\,(\Gamma_h c\ \text{grad}\ T)$$
$$+\ \text{div}\sum_{\text{all}\ \ell}\Gamma_\ell h_\ell\ \text{grad}\ m_\ell - \text{div}\ \vec{W}_s + S_{rad} + \ldots.$$

- ALTERNATIVE FORM:

$$\boxed{\rho\frac{D\tilde{h}}{Dt} = \text{div}\,(\Gamma_h\ \text{grad}\ \tilde{h}) + S_h}\ ,$$

WHERE $S_h \equiv \frac{\partial p}{\partial t} + S_{rad} + \ldots + \text{div}\,\{\Gamma_h\ \text{grad}\ \frac{\vec{u}^2}{2} - \vec{W}_s\}$
$$+\ \text{div}\,\{(\Gamma_h c - \sum_{\text{all}\ \ell}\Gamma_\ell m_\ell c_\ell)\ \text{grad}\ T$$
$$+\sum_{\text{all}\ \ell}(\Gamma_\ell - \Gamma_h)\,h_\ell\ \text{grad}\ m_\ell\}$$

HTE 1 / 5	12 / 15	ENERGY "CONSERVATION"; CARTESIAN-TENSOR FORM.

- DIFFERENTIAL EQUATION: $\frac{\partial}{\partial t}(\rho\tilde{h}) + \frac{\partial}{\partial x_i}(\rho u_i\tilde{h}) = \frac{\partial p}{\partial t}$

$$+\frac{\partial}{\partial x_i}\{\Gamma_h\, c\frac{\partial T}{\partial x_i} + \sum_{\text{all}\ \ell}\Gamma_\ell h_\ell\frac{\partial m_\ell}{\partial x_i}\} - \text{div}\ \vec{W}_s + S_{rad} + \ldots.$$

- SIMPLIFICATION: • $dh \equiv \sum_{\text{all}\ \ell} d(m_\ell h_\ell) = \sum_{\text{all}\ \ell} m_\ell dh_\ell + h_\ell dm_\ell$

$$= \sum_{\text{all}\ \ell}(m_\ell c_\ell dT + h_\ell dm_\ell).$$

- ALSO $c \equiv \sum_{\text{all}\ \ell} m_\ell c_\ell$, SO THAT $c\,dT \equiv \sum_{\text{all}\ \ell} m_\ell c_\ell dT$.
- OFTEN $-\text{div}\ \vec{W}_s \approx \frac{\partial}{\partial x_i}\{\frac{\mu}{2}\frac{\partial}{\partial x_i}(u_j u_j)\}$.
- THEN RHS $\rightarrow \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}\sum_{\text{all}\ \ell}\{(\Gamma_\ell - \Gamma_h)h_\ell\frac{\partial m_\ell}{\partial x_i} + (\mu - \Gamma_h)\frac{\partial}{\partial x_i}(\frac{u_j u_j}{2})\} + S_{rad} + \ldots$

HTE 1 5	13 15	ENERGY "CONSERVATION"; DISCUSSION.

- IN STEADY FLOW, $\partial p/\partial t$ VANISHES.
- WHEN $\Gamma_\ell = \Gamma_h$, THE $\partial m_\ell/\partial x_i$ TERM VANISHES.
- WHEN $\mu = \Gamma_h$, THE $\partial(u_j u_j/2)/\partial x_i$ TERM VANISHES.
- OFTEN, S_{rad} IS SMALL.
- THE EQUALITY OF Γ_ℓ AND Γ_h HOLDS FOR ALL TURBULENT FLUIDS, AND SOME LAMINAR GASES.
- THE EQUALITY OF μ AND Γ_h NEARLY HOLDS FOR THE SAME CASES.
- THEREFORE S_h IS OFTEN SMALL, AND CAN SOMETIMES BE TAKEN AS ZERO.

HTE 1 5	14 15	GENERAL FORM OF "CONSERVATION" EQUATION.

- DEFINITION: LET ϕ STAND FOR ANY OF THE DEPENDENT VARIABLES m_ℓ, $\tilde{h}$, u_i, m_α, ...
- EQUATION: THE RELEVANT CONSERVATION EQUATION CAN ALWAYS BE WRITTEN AS:

$$\frac{\partial}{\partial t}(\rho\phi) + \text{div}\,(\rho\vec{u}\,\phi) = \text{div}\,(\Gamma_\phi\,\text{grad}\,\phi) + S_\phi$$

OR ITS CARTESIAN-TENSOR EQUIVALENT.

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{\partial}{\partial x_i}(\rho u_i\phi) = \frac{\partial}{\partial x_i}\left(\Gamma_\phi\frac{\partial\phi}{\partial x_i}\right) + S_\phi$$

HTE 1 5	15/15	CONCLUDING REMARKS ABOUT THE DIFFERENTIAL EQUATIONS

- THE FUNDAMENTAL PHYSICAL LAWS RELEVANT TO FLUID MECHANICS AND HEAT AND MASS TRANSFER HAVE BEEN REVIEWED.
- THEY HAVE LED TO SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS.
  - THIS CONCLUDES PART I OF THE LECTURES.
  - LATER CONSIDERATION MUST BE GIVEN TO HOW TO SOLVE THE EQUATIONS.
  - FIRST, HOWEVER, FURTHER SIMPLIFICATIONS WILL BE REQUIRED. THESE ARE IN PART II.

HTE 1 6	$\frac{1}{15}$	PART II. IDEALISATIONS. LECTURE 6. IDEAL FLUIDS.

CONTENTS:

- UNIFORM-PROPERTY INCOMPRESSIBLE VISCOUS FLUID.
- THE SAME AT LOW REYNOLDS NUMBER.
- UNIFORM-PROPERTY INCOMPRESSIBLE INVISCID FLUID.
- THE SAME, IRROTATIONAL.
- THE HIGHLY-RESISTED FLUID.

NOTE: THIS LECTURE MAKES CONNEXIONS WITH BRANCHES OF CLASSICAL FLUID MECHANICS.

HTE 1 6	$\frac{2}{15}$	UNIFORM-PROPERTY INCOMPRESSIBLE VISCOUS FLUID, WITHOUT SOURCES EXCEPT grad p.

- CONTINUITY: $\mathrm{div}\,\vec{u} = 0.$
- MOMENTUM : $\frac{D}{Dt} u_m = -\vec{i}_m . \mathrm{grad} \left(\frac{p}{\rho}\right) + \frac{\mu}{\rho} \mathrm{div\,grad}\, u_m.$
- SPECIES : $\frac{D}{Dt} m_\ell = \frac{\Gamma_\ell}{\rho} \mathrm{div\,grad}\, m_\ell,$
- ENERGY : $\frac{D}{Dt} \tilde{h} = \frac{\Gamma_h}{\rho} \mathrm{div\,grad}\, \tilde{h},$
- NOTES:
  - μ/ρ CAN BE WRITTEN AS ν, THE KINEMATIC VISCOSITY.
  - SOMETIMES A NEW SYMBOL IS USED FOR p/ρ.
  - Γ_ℓ/ρ AND Γ_h/ρ, ARE OFTEN REPLACED BY ν/σ_ℓ, ν/σ_h WHERE THE σ'S ARE PRANDTL/SCHMIDT NUMBERS.
  - WITHIN THIS IDEALISATION, $\tilde{h}$ CAN BE REPLACED BY h OR T.

HTE 1 6	3 15	DISCUSSION OF THE FLUID OF PANEL 2

- IN TWO DIMENSIONS, DIFFERENTIATION OF THE MOMENTUM EQUATION LEADS TO:

$\frac{D}{Dt}\omega = \nu \text{ div grad } \omega$, WHERE $\omega \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$, THE VORTICITY.

- THEN THE CONTINUITY EQUATION REDUCES TO:

$\text{div grad } \psi = -\omega$, WHERE $\psi \equiv$ STREAM FUNCTION, DEFINED BY:

$$u_1 \equiv \frac{\partial \psi}{\partial x_2},\ u_2 = -\frac{\partial \psi}{\partial x_1}$$

- SINCE THE FIRST (VORTICITY TRANSPORT) EQUATION IS FREE FROM PRESSURE, AND LIKE THAT FOR m_ℓ, IT IS CONVENIENT FOR CALCULATION.

- THE $\omega \sim \psi$ SYSTEM HAS BEEN EXTENSIVELY USED FOR PREDICTION PROCEDURES, BUT IS NOW OUTMODED.

HTE 1 6	4 15	UNIFORM-PROPERTY VISCOUS FLUID AT LOW REYNOLDS NUMBERS; EQUATIONS.

- SPECIAL FEATURE: CONVECTION ($\vec{u}$.grad) TERMS ARE NEGLIGIBLE COMPARED WITH OTHERS.

- CONTINUITY: NO CHANGE.

- MOMENTUM: $\frac{\partial}{\partial t} u_m = -\vec{i}_m \text{ grad } \frac{p}{\rho} + \frac{\mu}{\rho} \text{ div grad } u_m$,

- SPECIES, ENERGY, VORTICITY (2D): $\frac{\partial \phi}{\partial t} = \frac{\nu}{\sigma_\phi} \text{ div grad } \phi$.

NOTES: • THE ABSENCE OF CONVECTION INCREASES THE LINEARITY OF THE EQUATIONS AND ENHANCES THE POSSIBILITY OF ANALYTICAL SOLUTION.

- LUBRICATION THEORY USES THESE EQUATIONS.

HTE 1 6	$\frac{5}{15}$	UNIFORM-PROPERTY INCOMPRESSIBLE INVISCID FLUID EQUATIONS.

- CONTINUITY: $\mathrm{div}\ \vec{u} = 0$.
- MOMENTUM: $\frac{D}{Dt} u_m = -\vec{I}_m \cdot \mathrm{grad}\ (\frac{p}{\rho})$.
- SPECIES, ENERGY, VORTICITY (2D): $\frac{D}{Dt} \phi = 0$.

NOTE: • IT HAS BEEN PRESUMED THAT <u>ALL</u> TRANSPORT PROPERTIES ARE ZERO.

- FOR 3D FLOW, THE RHS OF THE VORTICITY EQUATION IS NOT ZERO; VORTICITY INCREASES AS A RESULT OF "STREAM-TUBE NARROWING".

HTE 1 6	$\frac{6}{15}$	UNIFORM-PROPERTY INCOMPRESSIBLE INVISCID FLUID; DISCUSSION.

- $\frac{D\phi}{Dt} = 0$ MEANS THAT THE FLUID PARTICLES REMAIN OF CONSTANT ϕ AS THEY TRAVEL.
- IN 3D FLOW, THERE ARE 3 VORTICITY COMPONENTS; AND, EVEN FOR INVISCID FLOW, $D\omega_i/Dt \neq 0$; HOWEVER, THE RIGHT-HAND SIDE IS A FUNCTION OF THE ω'S, WHICH FALLS TO ZERO WHEN ALL ω'S ARE ZERO.
- THEREFORE, A FLUID WHICH IS AT FIRST WITHOUT VORTICITY (I.E. IRROTATIONAL) REMAINS WITHOUT IT, IN BOTH 2D AND 3D FLOW.
- THIS CONDITION CAN OFTEN BE REGARDED AS BEING SUFFICIENTLY SATISFIED BY THE AIR OR WATER THROUGH WHICH AN AIRPLANE OR SHIP TRAVELS

HTE 1 6	$\frac{7}{15}$	UNIFORM INCOMPRESSIBLE IRROTATIONAL FLUID; THE VELOCITY POTENTIAL.

- DEFINITION: LET $\vec{u} = \text{grad}\ \phi$, WHERE ϕ IS A SCALAR VELOCITY POTENTIAL.
- CONDITION: FOR ϕ TO BE A SCALAR,

  $$\frac{\partial^2 \phi}{\partial x_1\, \partial x_2} = \frac{\partial^2 \phi}{\partial x_2 \partial x_1} \quad , \text{ETC.}$$

- SIGNIFICANCE: THIS IMPLIES $\frac{\partial}{\partial x_1} u_2 = \frac{\partial}{\partial x_2} u_1$, ETC.

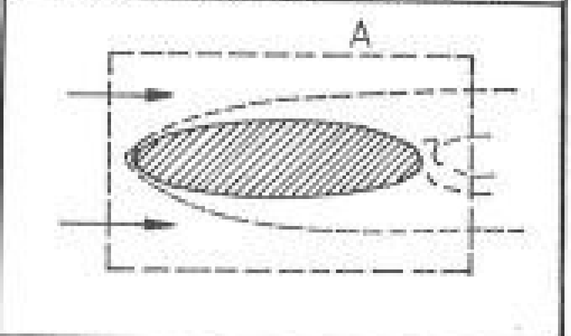
  I.E. $\omega_3 = 0$, ETC.; ALL THE VORTICITY COMPONENTS ARE ZERO.
- RESULT: THE FLUID IS IRROTATIONAL.

HTE 1 6	$\frac{8}{15}$	IRROTATIONAL FLOW; THE DIFFERENTIAL EQUATION OF ϕ.

- CONTINUITY EQUATION: $\text{div}\ \vec{u} = 0$.
- CONSEQUENCE: $\boxed{\text{div grad}\ \phi = 0}$.
- CARTESIAN TENSOR FORM: $\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_i} \phi = 0$.
- NAME: LAPLACE'S EQUATION.
- ANALOGOUS EQUATIONS:
  - STEADY-STATE FORM OF ϕ EQUATION ON PANEL 4.
  - CONSERVATION OF ELECTRIC CURRENT, WITH ϕ = ELECTRICAL POTENTIAL.
  - RESISTED-FLOW EQUATION (PANEL 11 BELOW).

HTE 1 6	9/15	IRROTATIONAL FLOW; RELEVANCE TO PRACTICAL FLOW SITUATIONS.

- NEAR SOLID SURFACES, VISCOUS EFFECTS CAUSE FLUID TO BE ROTATIONAL.
- FAR FROM THESE SURFACES, THE FLUID CAN OFTEN BE TAKEN AS IRROTATIONAL, E.G. IN ATMOSPHERE OR OCEAN.
- NUMERICAL ANALYSIS IS NEEDED FOR NEAR FIELD; BUT COMPUTER STORAGE FORBIDS EXTENSION OF FINITE-DIFFERENCE GRID FAR INTO IRROTATIONAL FIELD.
- POTENTIAL-FLOW THEORY IS USED TO GIVE BOUNDARY CONDITION AT GRID EDGE, A.



HTE 1 6	10/15	IRROTATIONAL-FLOW THEORY; DISCUSSION.

- THE EQUATION $\mathrm{div\ grad}\ \Phi = 0$ IS LINEAR.
  THEREFORE, IF $\Phi_1(x,y,z)$ AND $\Phi_2(x,y,z)$ ARE BOTH SOLUTIONS, SO IS $a\,\Phi_1 + b\,\Phi_2 + c$.
- THIS MEANS THAT THE PRINCIPLE OF SUPERPOSITION APPLIES.
- PARTICULARLY SIMPLE SOLUTIONS HAVE THE FORM $\Phi = \mathrm{const}/r^2$, WHERE $r \equiv$ DISTANCE FROM A "POINT-SOURCE".
- POTENTIAL-FLOW THEORY OFTEN INVOLVES FINDING THE POINT-SOURCE DISTRIBUTION IN SPACE WHICH WILL FIT GIVEN BOUNDARY CONDITIONS.

HTE 1 6	11 15	HIGHLY-RESISTED FLOW; DESCRIPTION.

- OCCURRENCE: FLOW THROUGH A POROUS ROCK, OR THROUGH A MATRIX OF CLOSELY-SPACED HEAT-EXCHANGER TUBES OR CATALYST PELLETS, ETC.
- MOMENTUM EQUATION (LECTURE 5, PANEL 5):
  $\frac{\partial p}{\partial x_i} = - F_i u_i$ WHERE $F_i \equiv f_i/u_i$.
- EXPLANATION OF F_i:
  - ITS NATURE IS: RESISTANCE COEFFICIENT PER UNIT LENGTH.
  - IT MAY VARY WITH u_i (NON-LINEAR RESISTANCE).
  - IT MAY VARY WITH DIRECTION (E.G. $F_2 > F_1$ WHEN THE RESISTANCE HAS DIRECTIONAL FEATURES.

HTE 1 6	12 15	HIGHLY-RESISTED FLOW; COMBINATION OF CONTINUITY AND MOMENTUM.

- CONTINUITY EQUATION (LECTURE 2, PANEL 3):
  $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0.$
- INSERTION OF u_i FROM PANEL 11:
  $\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x_i} ((\rho/F_i) \frac{\partial p}{\partial x_i})$

  OR $\frac{\partial \rho}{\partial t} = \mathrm{div}(C \text{ grad } p)$, WHERE $C \equiv \rho/F_i$.
- NOTES: • C CAN VARY WITH LOCATION AND WITH FLUID-FLOW DIRECTION.
  - IN A POROUS-MEDIUM FLOW, ρ AS MASS PER UNIT VOLUME MAY DIFFER FROM ρ AS FLUID DENSITY.

HTE 1 6	13 15	HIGHLY-RESISTED FLOW; DISCUSSION.

- FOR STEADY FLOW IN A UNIFORM MEDIUM (c = constant):
  div grad $p = 0$,
  SO p ACTS LIKE THE VELOCITY POTENTIAL.

- IF THE "BULK-CONTINUITY" FORM IS USED, THE EQUATION IS:
  $\mathrm{div}\,(\frac{1}{F}\,\mathrm{grad}\,p) = \frac{D\ell n\rho}{Dt}$.

- EVEN WHEN THE CONVECTION TERMS ARE NOT WHOLLY NEGLIGIBLE, IT MAY BE USEFUL TO CAST THE CONTINUITY EQUATION IN THIS FORM; THEN THE CONVECTION TERMS APPEAR AS "SOURCES" IN THE EQUATION.

HTE 1 6	14 15	PRESSURE AS A POTENTIAL; REMARKS FOR FUTURE REFERENCE.

- EQUATIONS SUCH AS: $\mathrm{div}\,(\frac{1}{F}\,\mathrm{grad}\,p)$ = non-zero RHS, ARE KNOWN AS POISSON EQUATIONS. THEY APPEAR, AS INDICATED ABOVE, IN HEAT-CONDUCTION, DIFFUSION, ELECTRICAL-FLOW, ETC., THEORY.

- MOST NUMERICAL PROCEDURES FOR SOLVING FLOW PROCESSES INVOLVE FINITE-DIFFERENCE FORMS OF A "POISSON EQUATION FOR PRESSURE".

- IN A PROCEDURE TO BE USED LATER, WHEN CONVECTION TERMS ARE SIGNIFICANT, THE POISSON EQUATION FOR PRESSURE <u>CORRECTION</u> APPEARS.

HTE 1 6	15/15	FINAL REMARKS

- CLASSICAL FLUID-DYNAMICAL AND HEAT CONDUCTION THEORY HAS CONCERNED ITSELF WITH SIMPLE EQUATIONS OF THE FORMS:

  $\mathrm{div\ grad}\ \Phi = 0, \quad \frac{D\Phi}{Dt} = 0, \quad \frac{D\Phi}{Dt} = \mathrm{const}\ \ \mathrm{div\ grad}\ \Phi,$

- NUMERICAL METHODS, ALTHOUGH CAPABLE OF SOLVING THE NON-SIMPLIFIED EQUATIONS, MAKE USE OF MANY CLASSICAL IDEAS AND TECHNIQUES.
- OTHER IDEALISATIONS ARE ALSO USEFUL, E.G. THAT DEFINED BY THE BOUSSINESQ APPROXIMATION, FOR WHICH ρ IS TREATED AS UNIFORM EXCEPT IN THE BODY FORCE TERM, WHERE IT IS TAKEN AS LINEAR IN, SAY, TEMPERATURE.

HTE 1 7	1/15	LECTURE 7. IDEALISATIONS OF CHEMICALLY-REACTING SYSTEMS.

CONTENTS:

- THE SIMPLE CHEMICALLY-REACTING SYSTEM.
- THE SCRS WITH FAST ATTAINMENT OF EQUILIBRIUM.
- RESULTING DIFFERENTIAL EQUATIONS.
- REACTEDNESS, τ.
- MIXTURE FRACTION f.

HTE 1 7	2/15	THE SIMPLE CHEMICALLY-REACTING SYSTEM; MOTIVATION.

- THE COMPLEXITY OF REAL COMBUSTION PROCESSES:
  - MOST FUELS PROCEED TO THEIR FINAL OXIDISED STATE BY WAY OF MANY INTERMEDIATES.
  - FOR FULL COMPUTATION, EACH CONCENTRATION OF INTERMEDIATE SPECIES (m_H, m_O, m_{CH_3}, m_{OH}, ETC.) MUST BE CALCULATED AT ALL POINTS.
  - THE ASSOCIATED COMPUTER TIME AND STORAGE ARE VERY LARGE; FOR 3D PROBLEMS THEY MAY BE PROHIBITIVE.
- THE UNIMPORTANCE OF KNOWING FULL DETAILS:
  - OFTEN ONLY THE MAJOR FEATURES ARE OF INTEREST (OUTLET TEMPERATURE, HEAT FLUX TO WALLS).

HTE 1 7	$\frac{3}{15}$	THE SCRS; DEFINITION.

1. THE MULTIPLE PATHS, REACTION PRODUCTS AND COMBINING RATIOS ARE REPLACED BY:

   1 kg fuel + s kg oxygen → (1+s) kg product,

   WHERE s (≡ STOICHIOMETRIC RATIO) IS A CONSTANT.

2. THE EXCHANGE COEFFICIENTS (Γ_{fu}, Γ_{ox}, Γ_{pr}) ARE EQUAL TO EACH OTHER, AND TO Γ_h, AT EACH POINT.

3. THE SPECIFIC HEATS OF ALL SPECIES ARE EQUAL TO EACH OTHER, AND INDEPENDENT OF T.

NOTE: • WITH LITTLE LOSS OF CONVENIENCE, c CAN BE ALLOWED TO DEPEND ON T ALONE.

HTE 1 7	$\frac{4}{15}$	THE SCRS; EXAMPLE.

REACTION: METHANE (CH_4) BURNS WITH OXYGEN.

1. $CH_4 + 2O_2 \rightarrow CO_2 + 2\,H_2O$; s = 4 kg/kg.
2. $\Gamma_{CH_4} = \Gamma_{O_2} = \Gamma_{CO_2} = \Gamma_{H_2O} = \lambda/c$, AT ALL POINTS.
3. $c_{CH_4} = c_{O_2} = c_{CO_2} = c_{H_2O} = c_{N_2}$ = constant.

NOTES: • (1) NEGLECTS CH_3, O, H, OH, HCO, ETC.

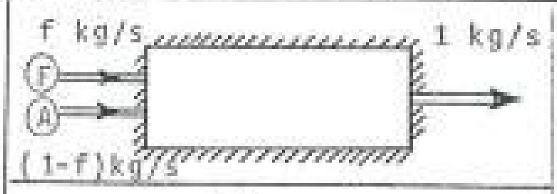
• (2) EQUALITY OF Γ'S IS NOT FAR FROM THE TRUTH FOR GASES, BUT WOULD BE A POOR APPROXIMATION FOR LIQUIDS (EXCEPT FOR TURBULENT FLOWS, WHERE EFFECTIVE Γ'S ARE IN QUESTION).

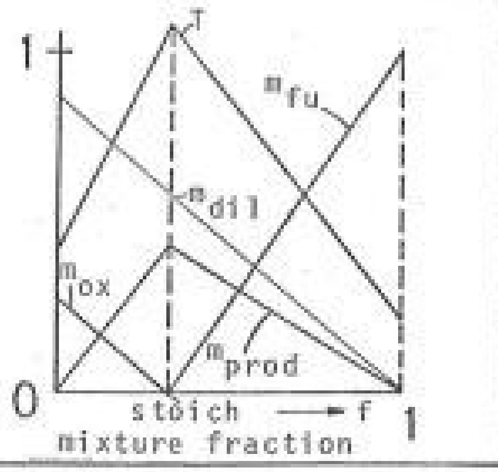
HTE 1 7	5 15	THE SCRS WITH RAPID ATTAINMENT OF CHEMICAL EQUILIBRIUM

- DEFINITION: THE REACTIVITY OF THE FUEL AND OXIDANT IS SUPPOSED TO BE SO GREAT THAT EITHER m_{fu} OR m_{ox} EQUALS ZERO AT EACH POINT.
- NOTES:
  - THIS SITUATION OFTEN PREVAILS, TO A CLOSE APPROXIMATION. THE FAST-REACTION ASSUMPTION IS THEREFORE PRACTICALLY USEFUL.
  - FAST-REACTION FLAMES ARE CALLED "PHYSICALLY CONTROLLED" OR "DIFFUSION CONTROLLED".
  - THE FAST-REACTION ASSUMPTION IS INDEPENDENT OF THE SCRS ASSUMPTIONS; I.E. EQUILIBRIUM CAN BE USEFULLY PRESUMED EVEN FOR COMPLEX MIXTURES.

HTE 1 7	6 15	THE FAST-REACTING SCRS IN A SIMPLE STEADY-FLOW PROCESS

- THE PROCESS: FUEL AND AIR MIX AND BURN IN A STEADY-FLOW ADIABATIC COMBUSTOR.
- CONSEQUENCES, FOR OUTLET-GAS STATE, OF:
  - SCRS ASSUMPTIONS;
  - FAST-REACTION ASSUMPTIONS;
  - CONSERVATION LAWS, ARE REPRESENTED IN THE SKETCH.→
- NOTES:
  - THE RELATIONS ARE LINEAR.
  - f ASSUMES IMPORTANCE LATER.





HTE 1 7	$\frac{7}{15}$	DIFFERENTIAL EQUATIONS FOR THE SCRS; EQUATION FOR $m_{fu} - m_{ox}/s$.

- STARTING POINTS:
  - LECTURE 5, PANEL 6, THE "m_ℓ EQUATION".
  - THE SCRS DEFINITION.
- PROCEDURE: DIVIDE THE m_{ox} EQUATION BY s, THE STOICHIOMETRIC CONSTANT, AND SUBTRACT FROM THE m_{fu} EQUATION.
- RESULT: $$\boxed{\frac{D}{Dt}\left(m_{fu} - \frac{m_{ox}}{s}\right) = \frac{1}{\rho}\,\mathrm{div}\left[\Gamma_{fu\,ox}\,\mathrm{grad}\left(m_{fu} - \frac{m_{ox}}{s}\right)\right]}$$
- COMMENTS:
  - $\Gamma_{fu\,ox} \equiv \Gamma_{fu} = \Gamma_{ox}$.
  - THIS IS A ZERO-SOURCE DIFFERENTIAL EQUATION.
  - SIMILAR EQUATIONS CAN BE DERIVED FOR:

    $m_{fu} + m_{prod}/(1 + s)$ AND $m_{ox}/s + m_{prod}/(1 + s)$.

HTE 1 7	$\frac{8}{15}$	DIFFERENTIAL EQUATIONS FOR THE SCRS; THE EQUATION FOR f.

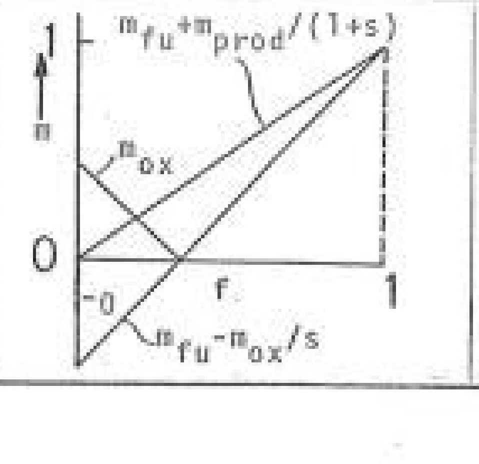
- GENERALISATION OF THE DEFINITION OF THE MIXTURE FRACTION, f:

  LET $f \equiv \dfrac{(m_{fu}-m_{ox}/s) - (m_{fu}-m_{ox}/s)_A}{(m_{fu}-m_{ox}/s)_B-(m_{fu}-m_{ox}/s)_A}$ WHERE A AND B DENOTE REFERENCE CONDITIONS, E.G. SUPPLY STATES.

- CONSEQUENCE, BECAUSE OF LINEARITY:

  $$\boxed{\frac{D}{Dt} f = \frac{1}{\rho}\,\mathrm{div}\left(\Gamma_{fu\,ox}\,\mathrm{grad}\, f\right).}$$

- NOTE: $f = m_{fu} + m_{prod}/(1 + s)$ IF A IS FUEL- AND PRODUCT-FREE, AND B IS PURE FUEL.



HTE 1 7	9/15	DIFFERENTIAL EQUATIONS FOR THE SCRS; COLLECTION.

- THE SOURCE-FREE EQUATION: $\frac{D}{Dt}\phi = \frac{1}{\rho}\,\mathrm{div}\,(\Gamma_\phi\,\mathrm{grad}\,\phi)$.
- VARIABLES SATISFYING THIS:
  - f, $(m_{fu} - m_{ox}/s)$, $(m_{fu} + m_{prod}/(1+s))$, $\left\{\frac{m_{ox}}{s} + \frac{m_{prod}}{(1+s)}\right\}$
  - m_α FROM LECTURE 5, PANEL 10.
  - m_{dil} (dil ≡ DILUENT, INERT BY DEFINITION).
- NOTES: • ALL SUCH ϕ'S ARE LINEARLY RELATED WHEN THE FLUID MIXTURES ARE FORMED FROM JUST TWO FLUID STREAMS A AND B (SEE PANEL 11 BELOW).
  - THEREFORE KNOWLEDGE OF ONE ϕ ALLOWS REMAINDER TO BE DEDUCED.

HTE 1 7	10/15	DIFFERENTIAL EQUATION FOR $\tilde{h}$ FOR EQUALITY OF EXCHANGE COEFFICIENTS.

- CONDITION: LET Γ_ℓ HAVE THE SAME VALUE AT EVERY POINT FOR ALL ℓ; AND LET THIS EQUAL Γ_h.
- NOTE: THE SCRS FULFILLS THIS CONDITION; BUT ONLY POINT (2) OF ITS DEFINITION IS NEEDED.
- CONSEQUENCE: $\frac{D}{Dt}\tilde{h} = \frac{1}{\rho}\{\mathrm{div}\,(\Gamma_h\,\mathrm{grad}\,\tilde{h}) + \mathrm{sources}\}$,

  WHERE "SOURCES" INCLUDES ONLY: $\frac{\partial p}{\partial t}$, s_{rad}, SHEAR WORK.
- EXPLANATION: IN THE s_h OF LECTURE 5, PANEL 11, THE FINAL div (....) VANISHES.
- COMMENT: IF THE SOURCES ARE ABSENT, THE $\tilde{h}$ EQUATION HAS THE FORM OF THOSE OF PANEL 9.

HTE 1 7	$\frac{11}{15}$	A GENERAL THEOREM CONCERNING SIMILAR DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS.

- PROPOSITION: LET ϕ_I AND ϕ_{II} BE TWO DISTINCT FLUID PROPERTIES, BOTH OBEYING: $\frac{D}{Dt}\phi = \text{div}\,(\Gamma_\phi\ \text{grad}\ \phi)$; AND LET THEIR VALUES BE SPECIFIED ONLY IN THE ENTERING A AND B STREAMS, THEIR NORMAL GRADIENTS AT ALL OTHER BOUNDARIES BEING ZERO.
- CONSEQUENCE: $\frac{\phi_I - \phi_{I,A}}{\phi_{I,B} - \phi_{I,A}} = \frac{\phi_{II} - \phi_{II,A}}{\phi_{II,B} - \phi_{II,A}}$ AT ALL POINTS.
- PROOF: • DEFINE ϕ_{III} AS DIFFERENCE OF THESE EXPRESSIONS.
  - ϕ_{III} SATISFIES THE EQUATION (CHECK BY SUBSTITUTION).
  - $\phi_{III} = 0$ IN A AND B STREAMS; GRADIENT = 0 ELSEWHERE ON BOUNDARIES.
  - HENCE THE SOLUTION MUST BE $\phi_{III} = 0$ THROUGHOUT.

HTE 1 7	$\frac{12}{15}$	CONSEQUENCES FOR RELATIONS BETWEEN f, $\tilde{h}$, m_α, m_{dil}.

- CONDITIONS: • TWO INLET STREAMS OF UNIFORM CONDITION.
  - GRADIENTS OF ALL VARIABLES NORMAL TO OTHER BOUNDARIES (WALLS, OUTFLOW) EQUAL TO ZERO (I.E. IMPERMEABLE ADIABATIC WALLS; SIMPLE OUTFLOW).
  - RADIATION, $\partial p/\partial t$ AND KINETIC HEATING NEGLIGIBLE.
- CONSEQUENCES: • f, $\tilde{h}$, m_α, m_{dil} ARE LINEARLY RELATED (SEE PANEL 9 ABOVE).
  - KNOWLEDGE OF ONE LEADS TO KNOWLEDGE OF ALL.
  - IF, FURTHER, FAST REACTION CAN BE PRESUMED, THE COMPLETE FLUID STATE IS KNOWN.

HTE 1 7	13/15	THE LINEAR $f \sim \tilde{h} \sim m_{\alpha} \sim m_{dil}$ RELATIONS; DISCUSSION.

- THE RELATIONS ARE NOT DEPENDENT ON THE VALIDITY OF THE SCRS ASSUMPTIONS, APART FROM THAT CONCERNING Γ'S.
- FOR TURBULENT FLOWS, WHERE EFFECTIVE Γ'S ARE APPROPRIATE, EQUALITY OF Γ'S IS CLOSELY ATTAINED.
- THE LINEAR RELATIONS ARE THEREFORE OFTEN USABLE, AND OFTEN USED, IN COMBUSTION PRACTICE.
- THE SOURCE OF ONE OF THE ϕ'S, FROM WHICH OTHERS MAY BE DEDUCED, MAY OF COURSE BE EXPERIMENTAL RATHER THAN A PREDICTION.

HTE 1 7	14/15	THE LINEAR $f \sim \tilde{h} \sim m_{\alpha} \sim m_{dil}$ RELATIONS; CONSEQUENCES FOR THE SCRS.

- FOR THE FAST-REACTING SCRS, THE DIAGRAM OF PANEL 6 PREVAILS.
- IF THE FULLY REACTED (BURNED) STATE IS DENOTED BY SUBSCRIPT b, $m_{ox,b}$, $m_{fu,b}$ ETC. CAN BE LINEARLY RELATED TO f; BUT THE CONSTANTS ARE DIFFERENT FOR $f < f_{stoich}$ AND $f > f_{stoich}$.
- LET THE "REACTEDNESS" τ BE DEFINED BY: $\tau \equiv (m_r - m_{r,u})/(m_{r,b} - m_{r,u})$ WHERE SUBSCRIPT u DENOTES THE UNREACTED (UNBURNED) STATE.
- THEN THE STATE OF THE FLUID CAN BE FULLY DEFINED BY SPECIFICATION OF TWO VARIABLES, f AND τ.

HTE 1 7	15 ― 15	FINAL REMARKS ABOUT IDEALISATIONS OF CHEMICALLY-REACTING SYSTEMS.

- THE FAST-REACTION ASSUMPTION AND THE LINEAR $\tilde{h} \sim f \sim$ ETC. RELATION ARE EXTREMELY USEFUL FOR THE UNDERSTANDING AND ANALYSIS OF GAS-TURBINE, DIESEL-ENGINE AND SOME FURNACE COMBUSTION PROCESSES.
- THE SCRS PERMITS COMPUTATIONS OF AN EXPLORATORY KIND TO BE PERFORMED SIMPLY.
- THE IDEA OF REACTEDNESS IS NOT NECESSARILY TIED TO THE SCRS; BUT A UNIQUE REACTION PATH (ALBEIT WITH MANY REACTANTS) IS ESSENTIAL.
- THE SCRS SUFFICES FOR MANY PRACTICAL COMPUTATIONS, BUT IS EASILY REFINED, E.G. BY ALLOWING: $c = c(T)$.

HTE 1 8	1/15	LECTURE 8. IDEALISATIONS OF TURBULENCE.

- CONTENTS:
  - THE IDEALISED TURBULENT FLUID.
  - TIME-AVERAGED CONSERVATION EQUATIONS.
    - CONTINUITY
    - MOMENTUM
    - CHEMICAL SPECIES
    - ENERGY
  - EQUATIONS FOR CORRELATIONS.
  - THE TURBULENCE-MODEL APPROACH.
- NOTE: THE SUBJECT MATTER OF THIS LECTURE IS DEALT WITH MORE EXTENSIVELY IN REPORT HTS/76/17.

HTE 1 8	2/15	DEFINITION OF THE IDEALISED TURBULENT FLUID

- THE IDEALISED TURBULENT FLUID IS SUPPOSED TO OBEY THE SAME EQUATIONS AS A GENERAL FLUID, EXCEPT THAT:
  - TIME-AVERAGED FLUID PROPERTIES (ϕ'S) APPEAR IN PLACE OF INSTANTANEOUS ONES.
  - TRANSPORT PROPERTIES (Γ'S) ARE AUGMENTED TO ACCOUNT FOR THE EFFECTS OF EDDY TRANSPORT.
  - SOURCE TERMS MAY REQUIRE MODIFICATION.
- NOTE: THE TIME OVER WHICH AVERAGING IS TO BE CONDUCTED IS LARGE COMPARED WITH FLUCTUATION TIME, BUT SMALL COMPARED WITH PHENOMENON TIME.

HTE 1 8	3/15	THE TIME-AVERAGED CONTINUITY EQUATION; DEFINITIONS.

- TIME-AVERAGING:
  - LET
    $\overline{\rho} \equiv [(\int_0^t \rho dt)/t]_{t \to large}$

    $\rho' \equiv \rho - \overline{\rho}$

    $\overline{u}_i \equiv [(\int_0^t u_i dt)/t]_{t \to large}$

    $u_i' \equiv u_i - \overline{u}_i$

    $\overline{\rho u_i} \equiv [(\int_0^t \rho u_i dt)/t]_{t \to large}$

    $\overline{\rho' u'} \equiv [(\int_0^t \rho' u_i' dt)/t]_{t \to large}$

- CONSEQUENCES: $\int_0^t \rho' dt$ and $\int_0^t u' dt \to 0$ as $t \to$ large;

  BUT $\overline{\rho' u_i'} = \overline{\rho u_i} - \overline{\rho}\, \overline{u}_i \neq 0$ in general.

HTE 1 8	4/15	TIME-AVERAGED CONTINUITY; STATEMENT.

- EQUATION: $\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\overline{\rho u_i}) = 0$
- ALTERNATIVE FORM: $\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i} (\overline{\rho}\, \overline{u}_i) = - \frac{\partial}{\partial x_i} (\overline{\rho' u_i'})$
- COMMENT:
  - IF THE DEFINITION OF THE IDEALISED TURBULENT FLUID IS FOLLOWED STRICTLY, THERE IS A "SOURCE" ON THE RHS.
  - OFTEN THIS SOURCE, I.E. THE DIFFERENCE BETWEEN $\overline{\rho u_i}$ AND $\overline{\rho}\, \overline{u}_i$, IS IGNORED.

HTE 1 8	5/15	THE TIME-AVERAGED CHEMICAL SPECIES CONSERVATION EQUATION.

- TIME-AVERAGING:

  LET $\overline{m}_\ell \equiv [(\int_0^t m_\ell \, dt)/t]_{t \to large}$,

  $m'_\ell \equiv m_\ell - \overline{m}_\ell$,

  $\overline{R}_\ell \equiv [(\int_0^t R_\ell \, dt)/t]_{t \to large}$,

  ETC.

- THE EQUATION:

$$\frac{\partial}{\partial t}(\overline{\rho}\ \overline{m}_\ell) + \frac{\partial}{\partial x_i}(\overline{\rho u_i}\ \overline{m}_\ell) = \frac{\partial}{\partial x_i}\left(\overline{\Gamma_\ell \frac{\partial m_\ell}{\partial x_i}} - \overline{(\rho u_i)' m_\ell'}\right) + \overline{R}_\ell - \frac{\partial}{\partial t}\overline{\rho' m_\ell'}\ ,$$

- NOTE: OFTEN $\overline{\rho}\ \overline{u}_i$ IS USED IN PLACE OF $\overline{\rho u_i}$.

HTE 1 8	6/15	TIME-AVERAGED CHEMICAL-SPECIES EQUATION; DISCUSSION.

- $\overline{\Gamma_\ell \frac{\partial m_\ell}{\partial x_i}}$ IS OFTEN TAKEN AS EQUAL TO $\overline{\Gamma_\ell}\ \frac{\partial \overline{m}_\ell}{\partial x_i}$
- $\overline{(\rho u_i)' m_\ell'}$ IS OFTEN WRITTEN AS: $-\Gamma_{t,\ell} \frac{\partial \overline{m}_\ell}{\partial x_i}$, WHERE $\Gamma_{t,\ell}$ IS DEFINED BY THIS EQUIVALENCE AND IS CALLED THE "TURBULENT EXCHANGE COEFFICIENT" OF SPECIES ℓ.
- $\overline{R}_\ell$ IS THE TIME-AVERAGE REACTION RATE. IT IS USUALLY QUITE DIFFERENT FROM $R(\overline{m}_\ell, \overline{m}_m, \overline{T} \ldots)$.
- THE TERM $\frac{\partial}{\partial t}\ \overline{\rho' m'_\ell}$ IS OFTEN IGNORED. IT IS PROBABLY SMALL, BECAUSE ρ' AND m'_ℓ ARE NOT LIKELY TO BE STRONGLY CORRELATED.

HTE 1 8	$\frac{7}{15}$	THE TIME-AVERAGED MOMENTUM-"CONSERVATION" EQUATION

- TIME-AVERAGING:

  LET $\overline{u_i}$ AND u'_i BE DEFINED AS BEFORE.
  LET $\overline{\rho u_j}$ AND $(\rho u_j)'$ BE SIMILARLY DEFINED.

- THE EQUATION:

$$\frac{\partial}{\partial t}(\bar{\rho}\,\bar{u}_i) + \frac{\partial}{\partial x_j}\left(\overline{\rho u_j}\;\overline{u_i} + \bar{p}\,\delta_{ij} - \overline{\mu\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i} - \frac{1}{3}\,e_{kk}\delta_{ij}\right]}\right)$$
$$= \overline{g_i} - \overline{f_i} - \frac{\partial}{\partial t}\,\overline{\rho' u'_i} - \frac{\partial}{\partial x_j}\,\overline{(\rho u_j)' u'_i}.$$

- IN FULL DETAIL, THE LAST TERM BECOMES (FOR m-DIRECTION MOMENTUM): $\frac{\partial}{\partial x_1}\,\overline{(\rho u_1)' u'_m} + \frac{\partial}{\partial x_2}\,\overline{(\rho u_2)' u'_m} + \frac{\partial}{\partial x_3}\,\overline{(\rho u_3)' u'_m}$

HTE 1 8	$\frac{8}{20}$	TIME-AVERAGED MOMENTUM EQUATION; DISCUSSION.

- USUALLY $\frac{\partial}{\partial t}\,\overline{\rho' u'_i}$ IS NEGLECTED ON THE GROUNDS THAT FLUCTUATIONS IN ρ AND u_i ARE UNCORRELATED.
- SIMILARLY, μ' IS SUPPOSED UNCORRELATED WITH u'_i ETC.
- THE QUANTITIES $\overline{(\rho u_j)' u_i'}$ ARE KNOWN AS THE REYNOLDS STRESSES.
- OFTEN, THE APPROXIMATION IS MADE:

$$-\frac{\partial}{\partial x_j}\,\overline{(\rho u_j)' u'_i} = \frac{\partial}{\partial x_j}\left[\mu_t\left\{\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} - \frac{1}{3}\,\bar{e}_{kk}\,\delta_{ij}\right\}\right],$$

  WITH μ_t AS A TURBULENT VISCOSITY.

- μ_t NEED NOT BE ISOTROPIC.

HTE 1 / 8	9 / 15	THE TIME-AVERAGED ENERGY-"CONSERVATION" EQUATION

- TIME-AVERAGING:

  LET $\bar{\tilde{h}} \equiv \left[\left(\int_o^t \tilde{h}\, dt \right) / t \right]_{t \to large}$

  $\tilde{h}' \equiv \tilde{h} - \bar{\tilde{h}}$.

  NOTE: $\bar{\tilde{h}} = \bar{h} + \frac{1}{2}(\bar{u}_i \bar{u}_i + \overline{u'_i u'_i})$

  WHERE $\frac{1}{2}\, \bar{u}_i \bar{u}_i \equiv$ KINETIC ENERGY OF MEAN MOTION,

  $\frac{1}{2}\, \overline{u'_i u'_i} \equiv$ KINETIC ENERGY OF FLUCTUATING MOTION, $\equiv k$.

- THE EQUATION: $\frac{\partial}{\partial t} (\bar{\rho}\, \bar{\tilde{h}}) + \frac{\partial}{\partial x_i} (\overline{\rho u_i}\, \bar{\tilde{h}}) = \frac{\partial}{\partial x_i} \left(\overline{\Gamma_h \frac{\partial \tilde{h}}{\partial x_i}} \right) + \bar{S}_h$

  $- \frac{\partial}{\partial t}\, \overline{\rho' \tilde{h}'} - \frac{\partial}{\partial x_i} \left(\overline{(\rho u_i)'\, \tilde{h}'} \right)$.

HTE 1 / 8	10 / 15	TIME-AVERAGED ENERGY EQUATION; DISCUSSION.

- USUALLY $\frac{\partial}{\partial t}\, \overline{\rho' \tilde{h}'}$, IS NEGLECTED.

- $\overline{\Gamma_h \frac{\partial \tilde{h}}{\partial x_i}}$ IS TAKEN AS $\bar{\Gamma}_h \frac{\partial \bar{\tilde{h}}}{\partial x_i}$.

- $-\overline{(\rho u_i)' \tilde{h}'}$ IS REPLACED BY $\Gamma_{h,t} \frac{\partial \bar{\tilde{h}}}{\partial x_i}$, WHERE $\Gamma_{h,t}$ IS THE TURBULENT EXCHANGE COEFFICIENT FOR HEAT.

- EXPRESSIONS WITHIN $\bar{S}_h$ REQUIRE DETAILED CONSIDERATION; BUT OFTEN THE WHOLE OF $\bar{S}_h$ IS SMALL.

HTE 1 8	11 15	THE GENERAL PROBLEM OF CALCULATING TURBULENT-FLOW PHENOMENA

- THE GENERAL FORM OF THE EQUATION IS SIMILAR TO THAT FOR LAMINAR FLOW. IT IS:

$$\frac{\partial}{\partial t} \overline{\rho}\, \overline{\phi} + \frac{\partial}{\partial x_i} (\overline{\rho}\, \overline{u}_i \overline{\phi}) = \frac{\partial}{\partial x_i} \left((\overline{\Gamma}_\phi + \Gamma_{\phi,t}) \frac{\partial \overline{\phi}}{\partial x_i} \right) + \overline{S}_\phi \quad .$$

- THE EQUATION IS AS EASY TO SOLVE AS ITS LAMINAR COUNTER-PART, SO LONG AS EXPRESSIONS ARE AVAILABLE FOR $\Gamma_{\phi,t}$, $\overline{S}_\phi$.
- NOTE THAT $\overline{\rho u_i}$ HAS BEEN REPLACED BY $\overline{\rho u_i}$ IN THE SECOND TERM. THIS CREATES THE FURTHER TERM $- \frac{\partial}{\partial x_i} (\overline{\rho' u_i}\ \overline{\phi})$ IN $\overline{S}_\phi$; BUT THIS MAY BE IGNORED.
- $\Gamma_{\phi,t} \frac{\partial \overline{\phi}}{\partial x_i}$ IS THE FORM POSTULATED FOR $- \overline{(\rho u_i)'\phi'}$

HTE 1 8	12 15	METHODS OF SOLVING THE GENERAL TURBULENT-FLOW PROBLEM: TURBULENCE MODELS.

- DEFINITION: A TURBULENCE MODEL IS A SET OF EQUATIONS WHICH, WHEN ADDED TO THE CONSERVATION EQUATIONS IN THEIR TIME-AVERAGED FORM, RENDERS THE MATHEMATICAL PROBLEM COMPLETE, I.E. SOLUBLE WHEN THE INITIAL AND BOUNDARY CONDITIONS ARE SUPPLIED.
- TYPES OF TURBULENCE MODEL:
  - THE TM MAY SUPPLY EXPRESSIONS FOR THE TURBULENT EXCHANGE COEFFICIENTS AND SOURCES (Γ_t's, $\overline{S}$'s) DIRECTLY; OR IT MAY SUPPLY THE CORRELATIONS THEY REPRESENT.
  - THE TM MAY INVOLVE ALGEBRAIC EQUATIONS ONLY; OR IT MAY INVOLVE ADDITIONAL "CONSERVATION"-TYPE EQUATIONS.

HTE 1 8	13 — 15	TURBULENCE MODELS: EXPRESSIONS FOR $\Gamma_{\phi,t}$

- TYPICAL FORM:

  $\Gamma_{\phi,t}$ = (density) x (mixing length) x (random velocity).
- ISOTROPIC FORM:

  $\Gamma_{\phi,t} = \bar{\rho}\ \ell\ k^{\frac{1}{2}}$,

  WHERE $k \equiv \frac{1}{2}(u_1{}^2 + u_2{}^2 + u_3{}^2)$, THE TURBULENCE ENERGY, AND ℓ IS A LOCAL "EDDY SIZE".
- NON-ISOTROPIC FORM:

  $\Gamma_{\phi,t,j} = \bar{\rho}\ \ell_j\ \overline{(u_j')^2}$

  WHERE j IS THE DIRECTION OF TRANSPORT, AND ℓ_j IS THE LENGTH APPROPRIATE TO IT.

HTE 1 8	14 — 15	TURBULENCE MODELS; EXPRESSIONS FOR ℓ, ETC

- ALGEBRAIC PRESUMPTIONS; EXAMPLES:
  - $\ell/Y = \lambda\{y/Y\}$, ($Y \equiv$ flow width); • ℓ = const. $k^{3/2}/\varepsilon$
  - $\sqrt{k}$ = constant. $\ell \left|\frac{\partial \bar{u}_1}{\partial x_2}\right|$;
  - $\sqrt{\overline{u_2{}'^2}}/\sqrt{k}$ = function $\left[\ell \left|\frac{\partial \bar{u}_1}{\partial x_2}\right| / \sqrt{k}\right]$.
- CONSERVATION EQUATIONS; EXAMPLES:

  $\frac{Dk}{Dt} = \frac{1}{\rho}$ div $(\Gamma_{k,t}$ grad $k) + \frac{1}{\rho} S_k - \varepsilon$;

  $\frac{D\varepsilon}{Dt} = \frac{1}{\rho}$ div $(\Gamma_{\varepsilon,t}$ grad $\varepsilon) + \frac{\varepsilon}{k}\left(\frac{C_1}{\rho} S_k - C_2\ \varepsilon\right)$;

  WITH μ_t = constant $\rho k^2/\varepsilon$, ETC.

  $\varepsilon \equiv$ dissipation rate of turbulence energy.

HTE 1 8	15/15	FINAL REMARKS ABOUT IDEALISATIONS OF TURBULENCE

- TURBULENCE MODELS ARE LIKE CHEMICAL-REACTION MODELS: THE TASK IS TO FIND THE SMALLEST NUMBER OF FLUID PROPERTIES, AND EQUATIONS DESCRIBING THEM, THAT ADEQUATELY DESCRIBE, AND PERMIT PREDICTIONS OF, FLOW PHENOMENA.

- THERE IS NO "BEST" TURBULENCE MODEL IN GENERAL; FITNESS FOR PURPOSE IS THE CRITERION OF MERIT.

- THE SUBJECT IS A NEW ONE; AND MANY QUESTIONS REMAIN TO BE EXPLORED, e.g.
  (1) OF ALL POSSIBLE TWO-EQUATION MODELS, WHICH POSSESS THE GREATEST GENERALITY.
  (2) WHAT ARE THE BEST WAYS OF REPRESENTING THE EFFECTS ON TURBULENCE OF:- • LOW REYNOLDS NUMBER; • COMPRESSIBILITY; • BUOYANCY; • CHEMICAL REACTION-

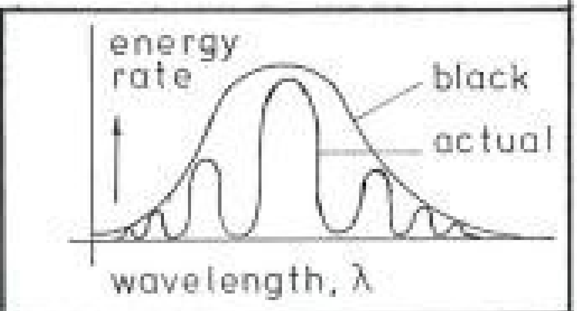
HTE 1 9	1/15	LECTURE 9. IDEALISATIONS OF RADIATION.

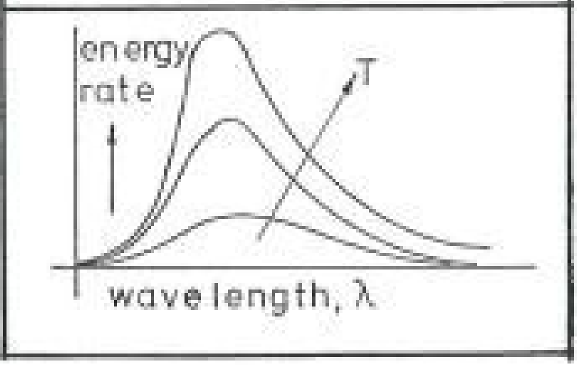
CONTENTS:

- SUBDIVISION WITH RESPECT TO WAVELENGTH.
- SIMPLIFICATION OF MATERIAL PROPERTIES.
- SIMPLIFICATION WITH RESPECT TO ANGULAR DISTRIBUTION:
  - TWO-FLUX MODEL;
  - FOUR-FLUX MODEL;
  - SIX-FLUX MODEL.
- THE ZONE METHOD.

HTE 1 9	2/15	VARIATION OF RADIATION WITH WAVELENGTH

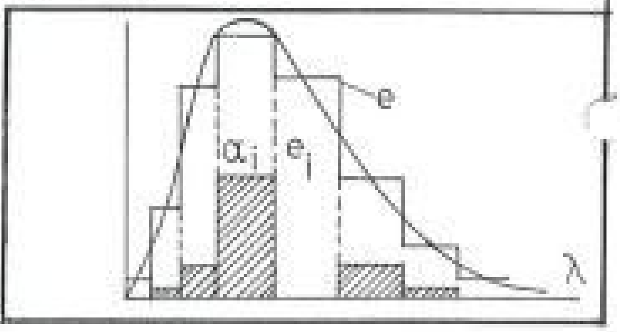
- BLACK BODIES EMIT AND ABSORB RADIATION IN DIFFERENT WAVELENGTHS ACCORDING TO A PRESCRIBED LAW (PLANCK'S): $e = e(\lambda, T)$.
- GREY BODIES EMIT AND ABSORB RADIATION SIMILARLY DISTRIBUTED: αe, WHERE α IS A CONSTANT.
- REAL MATERIALS EMIT AND ABSORB AT RATE $\alpha(\lambda, T) . e(\lambda, T)$.
- THE $\alpha(\lambda, T)$ FUNCTION IS COMPLICATED





HTE 1 9	3/15	SUB-DIVISION WITH RESPECT TO WAVELENGTH

- TO FACILITATE PRACTICAL COMPUTATION, THE WHOLE WAVELENGTH RANGE MAY BE SPLIT INTO FINITE-WIDTH INTERVALS, $\lambda_i \to \lambda_{i+1}$.
- THE BLACK-BODY INTENSITIES $e_{i+\frac{1}{2}}$ ARE KNOWN AVERAGES.
- ANY REAL MATERIAL IS CHARACTERISED BY A FINITE NUMBER OF "GREYNESS COEFFICIENTS" $\alpha_{i+\frac{1}{2}}(\lambda, T)$.
- THE COMPUTATION ATTENDS ONLY TO THE AVERAGE INTENSITIES IN THE INTERVALS.

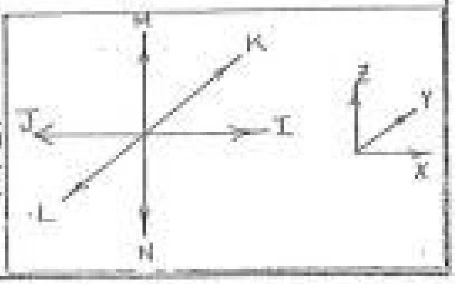


HTE 1 9	4/15	SIMPLIFICATION OF MATERIAL PROPERTIES

- $\alpha_{i+\frac{1}{2}}$ IS PUT EQUAL TO ZERO FOR THOSE INTERVALS IN WHICH IT IS VERY SMALL.
- THE NUMBER OF INTERVALS TAKEN IS SMALL.
- α'S ARE PRESUMED INDEPENDENT OF TEMPERATURE.
- MANY COMPONENTS OF MIXTURES ARE TAKEN AS FULLY TRANSPARENT (α's = 0).
- THE EFFECTS OF THE VARIOUS COMPONENTS OF A MIXTURE ARE TAKEN AS ADDITIVE.
- NOTE: OFTEN IGNORANCE ENFORCES THE SIMPLIFICATION; SOMETIMES IT IS EMPLOYED FOR ECONOMY.

HTE 1 9	5 15	SIMPLIFICATION WITH RESPECT TO ANGLE; FLUX METHODS.

- THE PROBLEM: • BECAUSE OF THE VARIATION WITH ANGLE, RADIATION INTENSITY VARIES WITH SIX INDEPENDENT VARIABLES (3 SPACE DIMENSIONS; WAVELENGTH; TWO DIRECTION COSINES).
  - ALLOWANCE FOR THE LAST TWO OVERBURDENS THE COMPUTATIONAL PROCEDURE.
- THE SOLUTION: • THE ANGULAR SPACE IS SUB-DIVIDED, BUT EVEN MORE DRASTICALLY THAN THE OTHER DIMENSIONS (DISTANCE; WAVELENGTH).
  - SPECIFICALLY, ATTENTION IS FOCUSSED ON: RADIATION FLUXES I, J CROSSING THE $y \sim z$ PLANE, K, L CROSSING THE $\sim z$ PLANE, ETC. • DIMENSIONS ARE w/m^2 (WAVELENGTH INTERVAL)



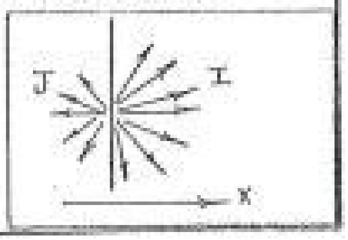
HTE 1 9	6 15	THE 1D TWO-FLUX METHOD; CARTESIAN GEOMETRY

- DIFFERENTIAL EQUATIONS (REF: PANEL 3.12)

$$\frac{dI}{dx} = -(a + s)I + aE + \frac{s}{2}(I + J),$$

$$\frac{dJ}{dx} = (a + s)J - aE - \frac{s}{2}(I + J),$$

- NOMENCLATURE:
  - $a \equiv$ ABSORPTIVITY (AND EMISSIVITY) PER UNIT LENGTH.
  - $s \equiv$ SCATTERING COEFFICIENT PER UNIT LENGTH.
  - $E \equiv$ BLACK BODY EMISSIVE POWER IN THE GIVEN WAVELENGTH INTERVAL.
- NOTES: • I IS TOTAL DIFFUSELY DISTRIBUTED RADIATION CROSSING CONST-x PLANE TO THE RIGHT.
  - J IS .... TO THE LEFT.



HTE 1 9	7/15	THE CARTESIAN TWO-FLUX METHOD; THE SECOND-ORDER EQUATION.

- MANIPULATION:

  ADDITION LEADS TO: $\frac{d(I+J)}{dx} = -(a+s)(I-J)$,

  SUBTRACTION LEADS TO: $\frac{d(I-J)}{dx} = -(a+s)(I+J)+2aE+s(I+J)$

  COMBINATION LEADS TO: $\boxed{\frac{d}{dx}\left(\frac{1}{a+s}\frac{d(I+J)}{dx}\right) = a(I+J-2E)}$

- NOTES: • $I-J$ IS THE NET ENERGY FLUX.
  - $I+J$ APPEARS AS THE ONLY TERM CONTAINING I OR J.
  - $1/(a+s)$ OCCUPIES THE EXCHANGE-COEFFICIENT POSITION.
  - THE EQUATION IS OF THE STANDARD FORM, WITH $D(I+J)/Dt = 0$, AND $S_{I+J} = a(2E-(I+J))$.
  - S_{I+J} EQUALS THE VOLUMETRIC ENERGY SINK.

HTE 1 9	8/15	THE CARTESIAN TWO-FLUX METHOD; THE CONDUCTION APPROXIMATION.

- WHEN a IS LARGE COMPARED WITH (FLOW DIMENSIONS)$^{-1}$ $I+J-2E$ MUST BE SMALL, SINCE S_{I+J} MUST BE FINITE.
- THEREFORE $I+J \approx 2E$ CAN BE INSERTED IN THE TOP EQUATION OF PANEL 7.
- WITH $Q_{rad} \equiv I-J$, THERE RESULTS:

$$Q_{rad} = -\frac{2}{a+s}\frac{dE}{dx},$$

- PUT $E = \sigma T^4$, WHICH IMPLIES THAT THE WHOLE WAVELENGTH RANGE IS CONSIDERED. THEN:

$$Q_{rad} = -\frac{8\sigma T^3}{a+s}\cdot\frac{dT}{dx}, \text{ I.E. } \boxed{\lambda_{rad} = \frac{8\sigma T^3}{a+s}}.$$

- $\sigma \equiv$ STEFAN-BOLTZMANN CONSTANT $= 5.680 \times 10^{-8}$ $J/m^2 s\ deg^4$.

HTE 1 9	9/15	THE TWO-FLUX METHOD; POLAR GEOMETRY

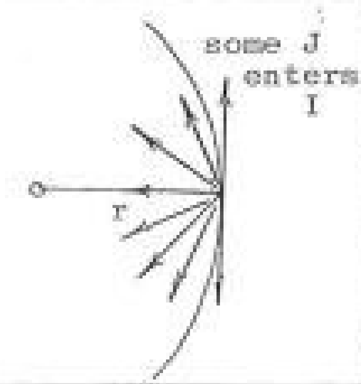
- DIFFERENTIAL EQUATIONS:

$$\frac{1}{r}\frac{d}{dr}(rI) = -(a+s)I + aE + \frac{s}{2}(I+J) + J/r ,$$

$$\frac{1}{r}\frac{d}{dr}(rJ) = (a+s)J - aE - \frac{s}{2}(I+J) + J/r ,$$

- NOMENCLATURE:

  $r \equiv$ RADIUS,
  $I \equiv$ OUTWARD-DIRECTED FLUX,
  $J \equiv$ INWARD-DIRECTED FLUX.



- NOTE: THE ASYMMETRY OF THE RELATIONS IS SURPRISING, BUT CORRECT: SOME OF J ENTERS I, BUT NO I ENTERS J.

HTE 1 9	10/15	THE POLAR-GEOMETRY TWO-FLUX FORMULATION; THE SECOND-ORDER EQUATION.

- MANIPULATION:

  ADDITION LEADS TO: $\frac{1}{r}\frac{d}{dr}\{r(I+J)\} = -(a+s)(I-J) + \frac{I+J}{r} - \frac{(I-J)}{r}$

  I.E. $\frac{d}{dr}(I+J) = -(a+s+1/r)(I-J)$,

  SUBTRACTION LEADS TO: $\frac{1}{r}\frac{d}{dr}\{r(I-J)\} = -a(I+J) + 2aE$

  COMBINATION LEADS TO: $\boxed{\frac{1}{r}\frac{d}{dr}\{\frac{r}{(a+s+1/r)}\frac{d(I+J)}{dr}\} = a(I+J-2E)}$

- NOTES: • $(I-J)$ IS STILL THE ENERGY FLUX.
  - $(I+J)$ IS STILL THE ONLY TERM CONTAINING I OR J.
  - $1/(a+s+1/r)$ OCCUPIES THE EXCHANGE-COEFFICIENT POSITION.
  - THE EQUATION REMAINS OF STANDARD FORM WITH MANY TERMS ABSENT.

HTE 1 / 9	11 / 15	THE FOUR-FLUX FORMULATION WITH POLAR GEOMETRY

- DIFFERENTIAL EQUATIONS:

$$\frac{1}{r}\frac{d}{dr}(rI) = -(a+s)I + aE + \frac{s}{4}(I+J+K+L) + J/r,$$

$$\frac{1}{r}\frac{d}{dr}(rJ) = (a+s)J - aE - \frac{s}{4}(I+J+K+L) + J/r,$$

$$\frac{dK}{dz} = -(a+s)K + aE + \frac{s}{4}(I+J+K+L),$$

$$\frac{dL}{dz} = (a+s)L - aE - \frac{s}{4}(I+J+K+L),$$

- NOMENCLATURE:
  - $r \equiv$ RADIAL DISTANCE;
  - $z \equiv$ AXIAL DISTANCE;
  - I, J ARE FLUXES IN +,- r DIRECTION;
  - K, L ARE FLUXES IN +,- z DIRECTION.

HTE 1 / 9	12 / 15	FOUR-FLUX POLAR GEOMETRY MODEL; SECOND-ORDER EQUATION.

- MANIPULATION: ADDITIONS AND SUBTRACTIONS AS BEFORE:
- RESULT:

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{(a+s+1/r)}\frac{d(I+J)}{dr}\right) = a(I+J-2E) + \frac{s}{2}(I+J-K-L)$$

$$\frac{d}{dz}\left(\frac{1}{(a+s)}\frac{d(K+L)}{dz}\right) = a(K+L-2E) + \frac{s}{2}(K+L-I-J)$$

- DISCUSSION:
  - THE TERM $s(I+J-K-L)$ REPRESENTS THE TRANSFER OF RADIATION FROM THE RADIAL TO THE AXIAL DIRECTION BY SCATTERING.
  - THE FLUXES AGAIN APPEAR ONLY IN THE PAIRED FORMS: $I+J$, $K+L$.
  - THE CONDUCTION APPROXIMATION ($I+J = 2E$) IS AGAIN VALID WHEN a IS LARGE, BECAUSE $a(I+J-2E)$ MUST BE FINITE.

HTE 1 9	13/15	SIX-FLUX POLAR GEOMETRY RADIATION MODEL

- DIFFERENTIAL EQUATIONS:

$$\frac{1}{r}\frac{d}{dr}(rI) = -(a+s)I + aE + \frac{s}{6}(I+J+K+L+M+N) + J/r$$

$$\frac{1}{r}\frac{d}{dr}(rJ) = (a+s)J - aE - \frac{s}{6}(I+J+K+L+M+N) + J/r$$

$$\frac{dK}{dz} = -(a+s)K + aE + \frac{s}{6}(I+J+K+L+M+N)$$

$$\frac{dL}{dz} = (a+s)L - aE - \frac{s}{6}(I+J+K+L+M+N)$$

$$\frac{1}{r}\frac{dM}{d\theta} = -(a+s)M + aE + \frac{s}{6}(I+J+K+L+M+N)$$

$$\frac{1}{r}\frac{dN}{d\theta} = (a+s)N - aE - \frac{s}{6}(I+J+K+L+M+N)$$

- NOTES: M AND N ARE INTENSITIES IN THE POSITIVE AND NEGATIVE θ DIRECTIONS.

HTE 1 9	14/15	SIX-FLUX POLAR-GEOMETRY SECOND-ORDER EQUATIONS.

- THE RESULT OF MANIPULATION IS:

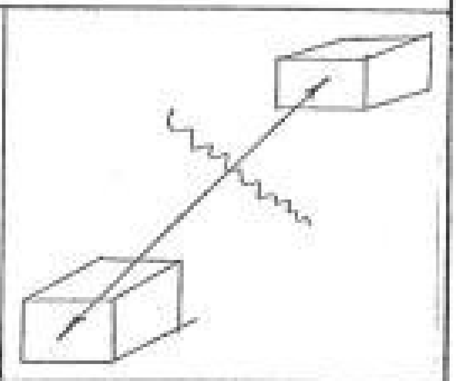
$$\frac{1}{r}\frac{d}{dr}\left\{\frac{r}{(a+s+1/r)}\frac{d(I+J)}{dr}\right\} = a(I+J-2E) + \frac{s}{3}\{2(I+J)-(K+L)-(M+N)\}$$

$$\frac{d}{dz}\left\{\frac{1}{(a+s)}\frac{d(K+L)}{dz}\right\} = a(K+L-2E) + \frac{s}{3}\{2(K+L)-(I+J)-(M+N)\}$$

$$\frac{1}{r}\frac{d}{d\theta}\left\{\frac{1}{(a+s)r}\frac{d(M+N)}{d\theta}\right\} = a(M+N-2E) + \frac{s}{3}\{2(M+N)-(I+J)-(K+L)\}$$

- DISCUSSION:
  - NOW RADIATION IS TRANSFERRED BY SCATTERING BETWEEN ALL THREE FLUX PAIRS.
  - THE EQUATIONS ARE EASY TO SOLVE NUMERICALLY.
  - APART FROM E, THEY ARE LINEAR; ANALYTICAL METHODS MAY BE USEFUL.

FINAL REMARKS



- AN ALTERNATIVE METHOD OF RADIATION CALCULATION IS THE HOTTEL "ZONE" METHOD. THIS INVOLVES A DIFFERENT SPATIAL AND ANGULAR SUBDIVISION OF THE RADIATION.
- THE ZONE METHOD IS IN PRINCIPLE MORE ACCURATE (WITH A LARGE NUMBER OF ZONES, THE EXACT SOLUTION MUST BE APPROACHED); BUT ITS COMPUTATIONAL EXPENSE IS VERY GREAT.
- THE FLUX FORMULATION CAN <u>PERHAPS</u> BE REFINED SO AS TO GIVE SOLUTIONS CONVERGING TO THE EXACT ONES; BUT CONCEPTUAL IMPROVEMENTS ARE NEEDED.

HTE 1 10	1/15	LECTURE 10. IDEALISATIONS OF MULTI-PHASE MIXTURES.

CONTENTS:

- THE GENERAL PROBLEM.
- THE NO-SLIP CONDITION.
- THE ONE-VARIABLE PARTICLE CLOUD.
- DIFFERENTIAL EQUATION FOR PARTICLE-NUMBER DISTRIBUTION
- PARTICLE-CHANGE LAWS.
- DIFFERENTIAL EQUATION FOR PARTICLE-MASS DISTRIBUTION
- FINITE-DIFFERENCING OF MASS DISTRIBUTION EQUATION
- TWO-VARIABLE PARTICLE CLOUDS

HTE 1 10	2/15	THE GENERAL PROBLEM OF FM AND HMT IN MULTI-PHASE MIXTURES; PRACTICAL OCCURRENCE.

- COMBUSTION: • FUEL IS OFTEN INJECTED AS A SPRAY OF DROPLETS, OR A CLOUD OF SOLID PARTICLES.
  - SOOT PARTICLES ARE FORMED IN FLAMES.
  - SOME OXIDES (E.G. Al_2O_3) ARE SOLID AT FLAME TEMPERATURES.
- STEAM GENERATORS: • STEAM-WATER MIXTURES ABOUND, AND FLOW IN MANY DIFFERENT REGIMES (DROPLET SUSPENSION, BUBBLY, FROTH, ETC.).
- NATURAL ENVIRONMENT: • SAND STORMS;
  - SEDIMENT TRANSPORT IN RIVERS.
- CHEMICAL ENGINEERING: • DISSOLUTION OF BUBBLES IN LIQUIDS;
  - DISTILLATION AND SEPARATION.

HTE 1 10	$\frac{3}{15}$	THE GENERAL PROBLEM; DESCRIPTION OF THE MULTIPHASE MIXTURE.

- DISTINGUISHING FEATURES: DISTINGUISHABLE COLLECTIONS OF MATTER WITHIN A VOLUME ELEMENT OF THE MIXTURE CAN DIFFER IN RESPECT OF:-
  - PHASE, • PARTICLE SIZE, • MOMENTUM PER UNIT MASS (3 COMPONENTS), • TEMPERATURE.
- INTERACTIONS: THESE DISTINGUISHABLE COLLECTIONS INTERACT IN CORRESPONDENCE WITH THE CHARACTERISTICS:-
  - PHASE CHANGE OCCURS (CONDENSATION, VAPORISATION, ETC.);
  - PARTICLE SIZES CHANGE AS A RESULT, AND ALSO BECAUSE OF COLLISION, FRAGMENTATION;
  - PARTICLE ~ FLUID DRAG EFFECTS MOMENTUM INTERCHANGE;
  - PARTICLES LOSE HEAT TO, AND GAIN IT FROM, THE SUSPENDING FLUID.

HTE 1 10	$\frac{4}{15}$	SOME IDEALISATIONS; A PROPERTY CORRELATED TWO CONTINUUM MODEL

DEFINITION:

- THE PARTICLE CLOUD AND THE SUSPENDING FLUID ARE REGARDED AS BOTH OCCUPYING THE SAME SPACE; EACH IS A CONTINUUM.
- THE VELOCITIES, TEMPERATURES, ETC., OF EACH CONTINUUM ARE DISTINCT; BUT THERE IS ONLY ONE VALUE OF EACH VARIABLE AT A POINT FOR EACH CONTINUUM (ALL PARTICLES HAVE THE SAME VELOCITY, DIFFERENT FROM THAT OF THE FLUID).

NOTES: • CARE IS NOW NEEDED TO DISTINGUISH THE MASS OF ONE FLUID PER UNIT VOLUME FROM ITS DENSITY.

- THE QUESTION OF HOW ALL PARTICLES ACHIEVE THE SAME VELOCITY, TEMPERATURE, ETC, IS NOT ANSWERED.

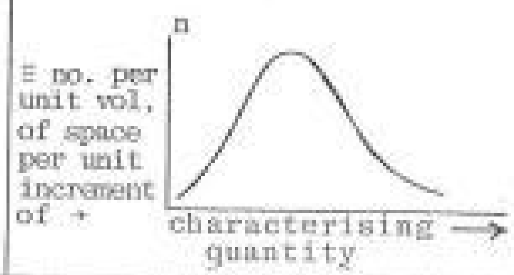
HTE 1 10	$\frac{5}{15}$	SOME IDEALISATIONS; THE PROPERTY-CORRELATED NO-SLIP PARTICLE CLOUD

DEFINITION:

- A CLOUD OF PARTICLES IS SUSPENDED IN A FLUID.
- AT ALL POINTS THE PARTICLE VELOCITY EQUALS THE FLUID VELOCITY IN MAGNITUDE AND DIRECTION.
- PARTICLES DIFFER FROM ONE ANOTHER IN RESPECT OF ONLY ONE PROPERTY, E.G. DIAMETER, TEMPERATURE (HENCE THE 1D IN THE TITLE).

NOTES:

- THE TASK IS THEN TO COMPUTE A SINGLE CURVE AT EACH POINT.
- THIS IDEALISATION IS USEFUL FOR MANY COMBUSTION PROBLEMS.



HTE 1 10	$\frac{6}{15}$	THE NO-SLIP PARTICLE CLOUD; AN EXAMPLE.

- DEFINITIONS:

  $n \equiv$ NO. OF PARTICLES PER UNIT VOLUME PER UNIT SIZE (s) INCREMENT.

  $s \equiv$ "SIZE" OF PARTICLE (LATER, $\equiv$ DIAMETER^2).

  $\vec{j} \equiv$ DIFFUSION-FLUX VECTOR OF PARTICLES ($no/m^2 s$).

  $\dot{s} \equiv$ RATE OF CHANGE OF PARTICLE SIZE, E.G. BY VAPORISATION.

- DIFFERENTIAL EQUATION:

$$\frac{\partial n}{\partial t} + \mathrm{div}\,(\vec{u}\,n + \vec{j}) = -\frac{\partial}{\partial s}(\dot{s}\,n)$$

- NOTE: $\dot{s}$ CAN BE INTERPRETED AS "VELOCITY OF A PARTICLE IN SIZE SPACE".

HTE 1 10	7 — 15	A SIMPLE SIZE-CHANGE LAW.

- PROCESS: VAPORISATION OF A SMALL LIQUID DROPLET INTO A SUSPENDING GAS.
- FORMULAE: $s \equiv D^2$, $D \equiv$ DROPLET DIAMETER,

  $$\dot{s} = -8\, \Gamma_h \ln \left(1 + c_{vap} \frac{(T_G - T_S)}{L}\right),$$

  WHERE Γ_h ≡ EXCHANGE COEFFICIENT OF GAS,
  c_{vap} ≡ SPECIFIC HEAT OF VAPOUR,
  L ≡ LATENT HEAT OF VAPORISATION,
  T_G, T_S ≡ TEMPERATURES OF GAS AND OF LIQUID SURFACE.
- NOTES: $\dot{s}$ IS INDEPENDENT OF s, IN THIS CASE; BUT MORE GENERAL CASES CAN BE CONSIDERED

HTE 1 10	8 — 15	THE 1D NO-SLIP MODEL; DIFFERENTIAL EQUATION IN TERMS OF MASS FRACTION.

- DEFINITION: LET m' ≡ MASS OF PARTICLE PER UNIT MASS OF LOCAL MIXTURE, PER s INCREMENT.
- CONSEQUENCE: $nM \equiv m'\rho$, WHERE
  M ≡ MASS OF A PARTICLE, $= M(s)$, ρ ≡ LOCAL-MIXTURE DENSITY

RESULTING DIFFERENTIAL EQUATION:

$$M^{-1}\left[\frac{\partial}{\partial t}(\rho m') + \mathrm{div}\,(\rho \vec{u} m' - \Gamma_p \,\mathrm{grad}\, m')\right] = -\frac{\partial}{\partial s}(\dot{s}\rho m'/M)$$

NOTES:
- M CAN BE DIVIDED THROUGH ON LHS BECAUSE M IS INDEPENDENT OF SPACE AND TIME.
- M CANNOT BE DIVIDED ON RHS BECAUSE M DEPENDS UPON s.
- Γ_p IS THE APPROPRIATE EXCHANGE COEFFICIENT FOR DROPLET INTERCHANGE.

HTE 1 10	9 15	DISCRETIZATION IN SIZE SPACE

- LET THE s RANGE BE BROKEN INTO A FINITE SET OF INTERVALS, NUMBERED: 1,2, ..., n........
- LET THE AVERAGE VALUES OF m' IN THE INTERVALS BE: m'_1, m'_2, ..., m'_n......, SO THAT:

$$m'_n \equiv (\int_{n_-}^{n_+} m'ds)/(s_+ - s_-);$$

  WHERE AND ARE THE INTERVAL BOUNDARIES
- IT IS NOW NECESSARY TO OBTAIN A DIFFERENTIAL EQUATION FOR m'_n BY INTEGRATING THAT FOR m' (PANEL 8).

HTE 1 11	10 15	THE DIFFERENTIAL EQUATION FOR m'_n.

- WITH $M \propto s^{3/2}$, THE RIGHT-HAND SIDE YIELDS:

$$\int_-^+ -M \frac{\partial}{\partial s}\left(\frac{\dot{s}\rho m'}{M}\right) ds = \rho\left((\dot{s}m')_- - (\dot{s}m')_+ + \frac{3}{2}\int_-^+ \frac{\dot{s}m'}{s} ds\right) .$$

- WHEN, AS IN THIS CASE, $\dot{s}$ DOES NOT DEPEND ON s, THE INTEGRAL EQUALS: $\rho\dot{s}(m'_- - m'_+ + \frac{3}{2} \int^+ (m'/s)ds)$
- THE PANEL-8 EQUATION THEREFORE BECOMES:

$$\frac{\partial}{\partial t}(\rho m'_n) + \text{div}\,(\rho\vec{u}\, m'_m - \Gamma \text{grad}\, m'_n) =$$
$$= \rho\{\dot{s}/(s_+ - s_-)\}\left[m'_- - m'_+ + \frac{3}{2}\int_-^+ (m'/s)\, ds.\right]$$

- EXPRESSIONS MUST NOW BE FOUND FOR m'_+ AND m'_-.

HTE 1 10	11 15	DETERMINATION OF THE INTERVAL-BOUNDARY VALUES, m'_+ AND m'_-

- m'_- AND m'_+ MUST BE CHOSEN ARBITRARILY BECAUSE THE DISCRETIZATION RESULTS IN INFORMATION-LOSS.
- WHEN $\dot{s}>0$, I.E. THE SIZE IS INCREASING WITH TIME, m'_- SHOULD BE TAKEN AS m'_{n-1}, AND m'_+ AS m'_n.
- THE REASON IS THAT, FOR PHYSICAL REASONS, THE CONTENTS OF THE SMALLER-SIZE INTERVAL CAN AFFECT THOSE OF THE LARGER-SIZE INTERVAL, BUT NOT VICE VERSA
- WHEN $\dot{s}<0$, m'_- SHOULD BE TAKEN AS m'_n, AND m_+ AS m_{n+1}, FOR THE SAME REASON.
- THESE CHOICES ARE AKIN TO THOSE EMPLOYED IN "UPWIND DIFFERENCING", (LECTURE 13, BELOW).
- TO TAKE $m'_- = \frac{1}{2}(m'_{n-1}+m'_n)$, SAY, WOULD LEAD TO PHYSICALLY UNREALISTIC RESULTS.

HTE 1 10	12 15	THE FINAL EQUATION FOR m'_n: THE CASE OF $\dot{s}<0$ (E.G. VAPORIZATION)

- FROM PANELS 10 AND 11, WITH AN OBVIOUS APPROXIMATION FOR THE INTEGRAL IN THE FORMER, AND WITH OTHER EASY CHANGES:

$$\rho\frac{Dm'_n}{Dt} = \mathrm{div}\,(\Gamma \,\mathrm{grad}\, m'_n) + \rho\dot{s}\quad \frac{m'_n - m'_{n+1}}{s_+ - s_-} + \frac{3m'_n}{s_+ + s_-}$$

- THERE ARE, OF COURSE, AS MANY EQUATIONS OF THIS TYPE AS THERE ARE INTERVALS.
- THE EQUATIONS ARE COUPLED VIA THE SOURCE TERMS.
- THE INFLUENCES ARE FROM LARGE n TO SMALL (FOR THIS CASE, VIZ. $\dot{s}<0$).

HTE 1 10	13/15	DISCRETE-GROUP MODELS; DISCUSSION.

- IN GENERAL, THE SOURCE TERM ON THE RHS OF THE DIFFERENTIAL EQUATION SHOULD BE AUGMENTED TO ACCOUNT FOR THE EFFECTS OF:-
  - DROPLET RUPTURE, WHICH DESTROYS LARGE DROPLETS AND CREATES SMALL ONES;
  - DROPLET COLLISION, WHICH CREATES LARGE ONES AT THE EXPENSE OF SMALL ONES.
  - BOUNDARY CONDITIONS REQUIRE SPECIAL ATTENTION FOR LIQUID PARTICLES, WHICH CAN ADHERE.
  - IN A LAMINAR FLUID, Γ WILL BE ALMOST ZERO; IN A TURBULENT ONE IT MAY VARY WITH s.

HTE 1 10	14/15	THE POSSIBILITY OF EMPLOYING MORE REALISTIC MODELS

- EXAMPLE: SUPPOSE THE DROPLETS WERE ALLOWED TO DIFFER IN RESPECT OF BOTH SIZE <u>AND</u> TEMPERATURE. HOW COULD THEIR DISTRIBUTIONS BE PREDICTED?
- SOLUTION: THE DROPLETS WOULD BE DIVIDED INTO GROUPS DISTINGUISHED BY BOTH s AND T. DIFFERENTIAL EQUATIONS FOR THE m_i's WOULD THEN BE MORE NUMEROUS; AND THE SOURCES WOULD EXHIBIT MORE LINKAGES.
- COMMENT: NO PROBLEM OF THIS KIND HAS BEEN SOLVED SO FAR. SOLUTION WOULD BE EXPENSIVE.

HTE 1 10	15/15	FINAL REMARKS ON IDEALISATIONS OF MULTI-PHASE MIXTURES.

- THE PROPERTY-CORRELATED NO-SLIP MODEL IS WELL-DEVELOPED AND UNDERSTOOD; BUT IT HAS STILL NOT BEEN EXTENSIVELY USED.
- THE 2-CONTINUUM MODEL IS CURRENTLY UNDER DEVELOPMENT AND INVESTIGATION; IT WILL BE USEFUL FOR GROUND-CLEARING COMPUTATIONS; BUT LACKS PHYSICAL REALISM IN SOME CASES.
- THE GENERAL PROBLEM IS STILL VERY FAR FROM SOLUTION.
- PHYSICAL KNOWLEDGE OF THE PROCESSES OF DROPLET BREAK-UP, PARTICLE COALESCENCE, ETC, IS FAR FROM ADEQUATE.

HTE 1 11	1 — 15	PART III. SPATIAL SUB-DIVISIONS. LECTURE 11. OVERALL BALANCES.

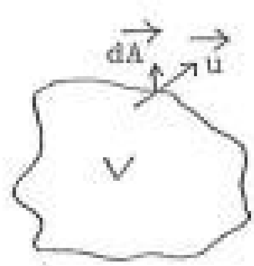
- CONTENTS:
  - THE GENERAL CONSERVATION EQUATION, DIRECT FORM.
  - THE GENERAL CONSERVATION EQUATION, "CORRECTION" FORM.
- APPLICATIONS TO:
  - CONTINUITY,
  - CHEMICAL SPECIES,
  - ENERGY,
  - OTHER EQUATIONS.
- IMPORTANCE:
  - PRACTICAL: OFTEN ONLY THE TERMS IN THE OVERALL BALANCE ARE OF INTEREST TO THE ENGINEER.
  - THEORETICAL: RATIONAL PREDICTION PROCEDURES SATISFY OVERALL BALANCES BEFORE MICRO-VOLUME ONES.

HTE 1 11	2 — 15	THE GENERAL CONSERVATION EQUATION FOR FLUID PROPERTY, ϕ.

- DIFFERENTIAL FORM: $\frac{\partial}{\partial t}(\rho\phi) + \text{div}(\rho\vec{u}\,\phi) = \text{div}(\Gamma_\phi \text{grad}\phi) + S_\phi$.
- OVERALL-BALANCE FORM:

$$\frac{\partial}{\partial t}\int_V \rho\phi dV + \int_A \phi\rho\vec{u}.d\vec{A} = \int_A \Gamma_\phi \text{grad}\phi.d\vec{A} + \int_V S_\phi\, dV$$

- NOMENCLATURE:
  - V ≡ VOLUME OF SPACE IN QUESTION, HAVING SURFACE AREA A;
  - $d\vec{A}$ ≡ AREA ELEMENT, DIRECTION BEING THAT OF THE NORMAL.
- REMARK: INTERNAL CONVECTION AND DIFFUSION FLUXES ARE WITHOUT IMPORTANCE.



HTE 1 11	$\frac{3}{15}$	GENERAL CONSERVATION EQUATION: APPLICATION TO A STIRRED TANK.

- DEFINITION OF STIRRED TANK: • ϕ UNIFORM;
  - ONE INLET AND ONE OUTLET FOR FLUID;
  - $(\Gamma_\phi \text{ grad } \phi)_A = g(\phi - \phi_{ext})$, WITH g UNIFORM.
- RESULT:

$$\boxed{\frac{d}{dt}(\rho\phi V) + \dot{m}_{out}\,\phi - \dot{m}_{in}\,\phi_{in} = Ag\,(\phi - \phi_{ext}) + S_\phi\,V}$$

- NOTES: • THE OUTGOING FLUID HAS THE SAME ϕ VALUE AS THE FLUID IN THE TANK (UNMIXING IS NOT POSSIBLE WITHOUT A SIEVE).
  - ρ, S_ϕ ARE ALSO UNIFORM.
  - FORMS WITH NON-UNIFORM g AND ϕ_{ext} CAN EASILY BE WRITTEN. THEN THE TERM $\int_A g(\phi - \phi_{ext})dA$ APPEARS.

HTE 1 11	$\frac{4}{15}$	GENERAL CONSERVATION EQUATION: CORRECTION FORM, STATEMENT.

- DEFINITION: • LET $\phi \equiv \phi_* + \phi'$, WHERE THE LATTER ARE ARBITRARILY DEFINED FUNCTIONS IN THE FIRST INSTANCE.
  - THEN LET ϕ' BE UNIFORM OVER V, EXCEPT THAT $\Gamma_\phi \text{ grad } \phi . d\vec{A} \equiv \Gamma_\phi \text{ grad } \phi_* . d\vec{A} + g\phi'\, dA$.
- EXPLANATION: • ϕ_*, VARYING WITH POSITION, MIGHT BE AN ESTIMATE OF THE TRUE ϕ DISTRIBUTION.
  - ϕ' MIGHT BE THE ANSWER TO: WHAT INCREMENT TO ϕ_*, UNIFORM, WOULD SATISFY THE OVERALL BALANCE EQUATION?
- THE EQUATION: $\frac{d}{dt}(\rho V \phi') + \int_A \phi' \rho \vec{u} . d\vec{A} - \int_A g\phi' dA - S'_\phi\, V =$

$$= -\left[\frac{\partial}{\partial t}\int_V \rho\phi_*\, dV + \int_A \phi_*\, \vec{u} . d\vec{A} - \int_A \Gamma_\phi \text{grad}\phi_* . d\vec{A} - \int_V S_\phi dV\right] .$$

HTE 1 / 11	5 / 15	GENERAL CONSERVATION EQUATION: CORRECTION FORM, DISCUSSION.

- THE RHS, -[.....], WHICH REPRESENTS THE UNBALANCE ASSOCIATED WITH ϕ_*, CAN BE CALLED THE "ERROR-SOURCE" $E_\phi V$.
- $S'_\phi V$ MUST BE DEFINED AS: $\left\{ \int_V \frac{\partial S_\phi}{\partial \phi} .dV \right\} \phi'$.
- NO ϕ'_{ext} APPEARS IN THE $\int g\ldots$ EXPRESSION, BECAUSE WE CAN EXPECT THAT ϕ_{ext} IS KNOWN; THERE IS NO POINT IN IMAGINING AN ERROR IN IT.
- WITH $\bar{g} \equiv (\int g \; dA)/A$, THE CORRECTION FORM BECOMES EVEN SIMPLER THAN THE STIRRED-TANK FORM:

$$\boxed{\frac{\partial}{\partial t}(\rho\phi' V) + \dot{m}_{out}\phi' - A\bar{g}\,\phi' - S'_\phi V = E_\phi V}$$

HTE 1 / 11	6 / 15	OVERALL CONTINUITY BALANCE EQUATION: DIRECT FORM.

- THE EQUATION: $\frac{\partial}{\partial t}\int_V \rho dV + \int_A \rho\vec{u}.d\vec{A} = \int_V S_{mass}\; dV$.
- NOTES:
  - THE SOURCE OF MASS MIGHT BE A SUPPLY THROUGH A SMALL PIPE, OR IN SOME OTHER WAY, THAT WE PREFER NOT TO INCLUDE IN $\int_A \rho\vec{u}.d\vec{A}$.
  - THE DIFFUSION TERM IS ABSENT.
  - THE STIRRED-TANK FORM IS SIMPLY:

    $\frac{d}{dt}(\rho V) + \dot{m}_{out} - \dot{m}_{in} = \bar{S}_{mass}\, V$ (often = 0).
  - THIS CAN BE USED TO ELIMINATE (SAY) $\dot{m}_{in}$ FROM THE ϕ EQUATION (PANEL 3).

HTE 1	7/15	OVERALL CONTINUITY BALANCE;
11		CORRECTION FORM, STATEMENT.

- PRELIMINARY NOTES: • $\phi = 1$ FOR CONTINUITY; SO THERE IS NO POINT IN CONSIDERING A CHANGE.
  - IT IS MORE USEFUL TO CONSIDER CHANGES TO ρ AND TO $\rho\vec{u}$.
- DEFINITIONS: • LET $\rho \equiv \rho_* + \rho'$; $\rho\vec{u}.d\vec{A} \equiv G dA$;
  - $G \equiv G_* + G'$; $S_{mass} \equiv S_* + S'$.
- EQUATION: $\frac{\partial}{\partial t}(\rho' V) + \int_A G' dA - \int_V S' dV$

$$= -\left[\frac{\partial}{\partial t}\int_V \rho_* dV + \int_A G_* dA - \int_V S_* dV\right]$$

$$\equiv E_{mass}\ V, \text{ SAY.}$$

- NOTE: FOR THE STIRRED TANK: $\frac{\partial}{\partial t}(\rho' V) + \dot{m}'_{out} - \dot{m}'_{in} + S'V = E_{mass}\ V$.

HTE 1	8/15	OVERALL CONTINUITY BALANCE;
11		CORRECTION FORM, DISCUSSION.

- USEFULNESS: • OFTEN A FLOW FIELD IS SPECIFIED WHICH SATISFIES CONTINUITY ONLY APPROXIMATELY ($E_{mass} \neq 0$).
  - THE FLOW FIELD MUST BE "CORRECTED" WITH A SMALL AMOUNT OF COMPUTATION.
  - THE PANEL-7 EQUATION PROVIDES A RELATION WHICH CORRECTIONS TO ρ, G AND S MUST OBEY.
- A FURTHER QUESTION: • THE EQUATION CONNECTS <u>SEVERAL</u> UNKNOWNS; <u>WHICH</u> SHOULD BE DETERMINED WITH ITS AID?
  - THIS QUESTION CANNOT BE SETTLED (EXCEPT ARBITRARILY) UNTIL SOME LINK HAS BEEN FOUND BETWEEN ρ', G' AND S'.

HTE 1 / 11	9 / 15	OVERALL CONTINUITY BALANCE; USE OF PRESSURE AS THE LINKING VARIABLE.

- PRESUMPTIONS: • LET ALL THE CORRECTIONS BE THE CONSEQUENCE OF A UNIFORM PRESSURE INCREMENT, p'.
  - LET THE RELATIONS BE LINEAR ONES:
    $\rho' \equiv a_\rho p'; \; G' = a_G p'; \; S' = a_S p'$.
  - LET THE a'S BE SUPPOSED KNOWN (OR KNOWABLE).
- CONSEQUENCE: $\boxed{\frac{\partial}{\partial t}(p' a V) + \int_A p' a_G dA + \int_V p' a_S dV = EV}$
- COMMENTS: • p', BEING UNIFORM, CAN BE TAKEN OUTSIDE THE TWO INTEGRALS.
  - THE RESULT IS AN EQUATION OF FORM: $\frac{dp'}{dt} + \alpha p' = \beta$.
  - THIS IS EASILY SOLVED.

HTE 1 / 11	10 / 15	OVERALL CHEMICAL-SPECIES BALANCE; CORRECTION FORM.

- DEFINITIONS: LET $\phi \equiv m_\ell$, $S_\phi \equiv R_\ell$.
- RESULT: $\boxed{\frac{\partial}{\partial t}(\rho m'_\ell V) + \dot{m}_{out} m'_\ell - A\bar{g} m'_\ell - \frac{\partial R_\ell}{\partial m_\ell} V m'_\ell = E_\ell V}$
- DISCUSSION: • IF $m_{*\ell}$ WERE DEFINED AS ZERO, $E_\ell V$ WOULD BE A LARGE QUANTITY; AND THE ORIGINAL (DIRECT) FORM OF CONSERVATION EQUATION WOULD BE RECOVERED.
  - THE USE OF THE ABOVE FORM IS MAINLY IN FINDING CORRECTIONS TO APPROXIMATELY CORRECT m_ℓ DISTRIBUTIONS.
  - NOTE THAT NO "LINKING VARIABLE" (LIKE p) HAD TO BE IMPORTED IN THIS CASE; m'_ℓ SERVES FOR THIS.

HTE 1 11	11/15	OVERALL CHEMICAL-SPECIES BALANCE; EXAMPLE OF USE OF CORRECTION FORM.

- THE PROBLEM: • FUEL AND AIR BURN IN A STEADY-FLOW FURNACE.
  - THE WALLS ARE IMPERMEABLE.
  - DIFFUSION FLUXES AT INLET AND OUTLET ARE NEGLIGIBLE.

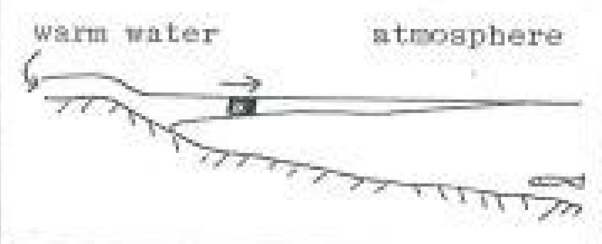
- THE VARIABLE: • CONSIDER THE MIXTURE FRACTION f, FOR WHICH A DISTRIBUTION f_* IS KNOWN. • DETERMINE THE UNIFORM f' WHICH, ADDED TO f, SATISFIES OVERALL CONSERVATION.
- EQUATIONS: $E_f\ V \equiv -\left[\dot{m}_{out}\ f_{*out} - \dot{m}_{fuel,in}\right]$,

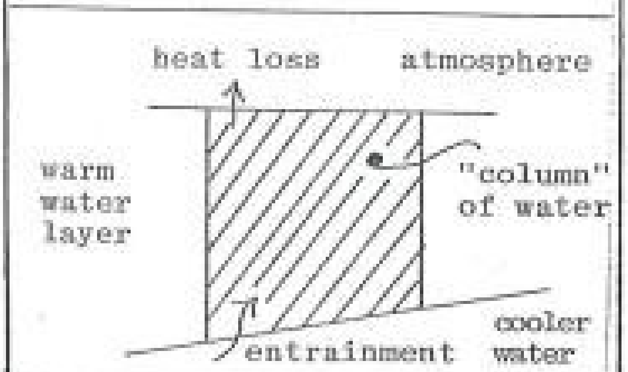
$$\boxed{\dot{m}_{out}\ f' = E_f V}$$

- REMARKS: • THERE ARE NO SOURCE OR TRANSIENT TERMS. • OF COURSE THE NEW f DISTRIBUTION IS SATISFACTORY ONLY OVERALL.

HTE 1 11	12/15	OVERALL ENERGY BALANCE EQUATION: APPLICATIONS OF THE DIRECT FORM.

- EXAMPLE: • WARM WATER, DISCHARGED ON TO A LAKE SURFACE, FLOWS OUTWARDS.
  - COHERENT "COLUMNS" OF WATER CAN BE IDENTIFIED.
  - TURBULENT MIXING ENSURES VERTICAL UNIFORMITY OF TEMPERATURE.
  - "ENTRAINMENT" OF COOLER FLUID MAY OCCUR FROM BELOW.
- NOTE: V INCREASES WITH TIME.





HTE 1 11	13 15	OVERALL ENERGY-BALANCE EQUATION; THE WATER-COLUMN EXAMPLE.

- EQUATION: $\frac{d}{dt}(\rho V c T) - \dot{m}_{ent} c\, T_{cool} = AU(T - T_{ext}) + S_{rad} V.$
- NOMENCLATURE: c ≡ SPECIFIC HEAT; $\dot{m}_{ent}$ ≡ ENTRAINMENT RATE; U ≡ HEAT TRANSFER COEFFICIENT; S_{rad} ≡ NET RADIATIVE INFLOW PER UNIT VOLUME.
- INTRODUCTION OF CONTINUITY EQUATION $\left(\frac{d}{dt}(\rho V) - \dot{m}_{ent} = 0\right)$

  LEADS TO:

$$\boxed{c\frac{dT}{dt} + \frac{\dot{m}_{ent}}{\rho V} c(T - T_{cool}) = \frac{AU}{\rho V}(T - T_{ext}) + \frac{S_{rad}}{\rho}}\ .$$

- COMMENT: • THIS LINEAR EQUATION IS EASY TO SOLVE.
  - LINEARITY DEPENDS ON U'S INDEPENDENCE OF T.

HTE 1 11	14 15	OVERALL ENERGY-BALANCE EQUATION; APPLICATION OF THE CORRECTION FORM.

- THE PROBLEM: • IN THE FURNACE OF PANEL 11, THE ESTIMATED h DISTRIBUTION ($h \equiv cT + H\, m_{fu}$, SAY) DOES NOT PRECISELY SATISFY THE OVERALL ENERGY-BALANCE EQUATION.
  - WHAT UNIFORM INCREMENT h' MUST BE APPLIED (LEADING TO A TEMPERATURE INCREMENT $T' \equiv h'/c$) TO PROCURE SATISFACTION?
- THE EQUATIONS: • $E_h\, V \equiv -[\dot{m}_{out}\, h_{+out} - \dot{m}_{air} h_{air} - \dot{m}_{fu} h_{fu} + \bar{Q}A - \bar{S}V].$
  - $\boxed{\dot{m}_{out}\, h' + A\, h'(\partial\bar{Q}/\partial h) - V\, h'\, \partial\bar{S}/\partial h = E_h V}$.
- NOTES: • THIS EQUATION ALLOWS h' TO BE DETERMINED.
  - $\partial\bar{Q}/\partial h$ AND $\partial\bar{S}/\partial h$ MAY INVOLVE LINEARISATION.
  - IN CASE OF NON-LINEARITY, ITERATION IS HELPFUL.

HTE 1 / 11	15 / 15	OVERALL-BALANCE EQUATIONS; FINAL REMARKS.

- OTHER EQUATIONS: OVERALL BALANCE EQUATIONS FOR MOMENTUM, RADIATION, PARTICLES, ETC., ARE ALSO USEFUL, IN BOTH DIRECT AND CORRECTION FORMS.
- BALANCES FOR SUB-DOMAINS: • THE WHOLE REGION OF INTEREST MAY BE DIVIDED INTO 2 PARTS.
  - THESE MAY ALSO BE SUB-DIVIDED.
  - WHEN MANY SUB-DOMAINS ARE CONSIDERED SEPARATELY, WE HAVE A "FINITE-DIFFERENCE" OR "FINITE-ELEMENT" SUB-DIVISION OF SPACE.
- IMPORTANCE OF "CORRECTION" FORM: • ALTHOUGH ALGEBRAICALLY EQUIVALENT TO THE DIRECT FORM, THE CORRECTION FORM HAS SPECIAL MERITS; ITS TERMS PERMIT SIMPLIFICATION WHEN ITERATIVE PROCEDURES ARE USED.

HTE 1 12	1/15	LECTURE 12. JETS, WAKES, PLUMES AND LAYERS.

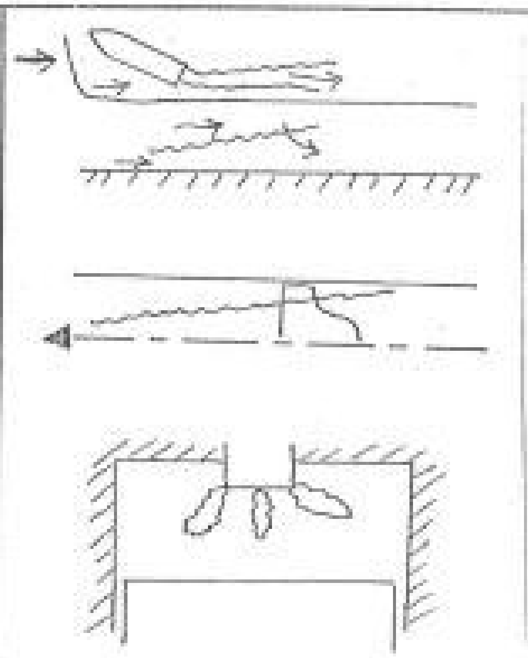
- CONTENTS:
  - QUALITATIVE DESCRIPTION.
  - PRACTICAL RELEVANCE.
  - MATHEMATICAL DESCRIPTION.
  - THE GENERAL CONSERVATION EQUATION.
  - SPECIAL FORMS.
  - PROFILE ASSUMPTIONS.
- NOTES: • JETS, WAKES, ETC., ARE PARTICULAR SUB-DOMAINS.
  - EVEN WHEN NUMERICAL METHODS ARE EMPLOYED, JET ANALYSES REMAIN OF GREAT VALUE.

HTE 1 12	2/15	QUALITATIVE DESCRIPTION OF JETS, WAKES, PLUMES AND LAYERS.

- JET: AN ELONGATED REGION, IN WHICH THE FLUID VELOCITY EXCEEDS THAT OF THE FLUID OUTSIDE IT, AND INTO WHICH FLUID IS DRAWN FROM OUTSIDE BY "ENTRAINMENT".
- WAKE: AS FOR JET, BUT WITH THE FLUID VELOCITY LOWER THAN THAT OF THE EXTERNAL FLUID.
- PLUME: AS FOR JET, BUT WITH SIGNIFICANT INFLUENCES OF BUOYANCY.
- LAYER: A HORIZONTALLY-EXTENDED FLUID REGION, LYING ABOVE DENSER OR BELOW LIGHTER FLUID, WITH WHICH HEAT, MASS AND MOMENTUM MAY BE EXCHANGED.

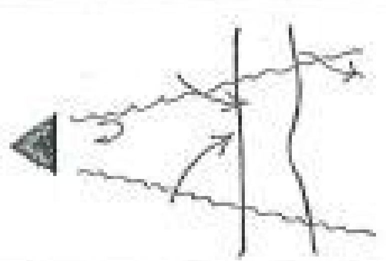
HTE 1 12	3 — 15	PRACTICAL RELEVANCE; EXAMPLES OF JETS.

- STEADY JETS: • ROCKET EXHAUST.
  - DILUTION AIR IN COMBUSTOR.
  - FILM COOLING OF COMBUSTOR WALL.
  - MIXING OF WARM WATER WITH COOLER RIVER WATER NEAR THE DISCHARGE POINT.
  - FLAME SPREAD BEHIND A BAFFLE IN A DUCT.
- UNSTEADY JETS: • INJECTION OF FUEL INTO DIESEL ENGINE CYLINDER.
  - FLOW IN "FLUIDICS" DEVICES.
  - SUDDEN OPERATION OF A STEAM WHISTLE.
- N.B.: SOME BOUNDARY LAYERS ON WALLS ARE JETS.



HTE 1 12	4 — 15	PRACTICAL RELEVANCE; EXAMPLES OF WAKES.

- STEADY WAKES: • THE FLOW BEHIND A TOWED MODEL SHIP (N.B. THAT BEHIND A SELF-PROPELLED SHIP IS PART WAKE AND PART JET; THE OVERALL MOMENTUM DIFFERENCE FROM THE EXTERNAL FLUID IS ZERO.).
  - THE FLOW BEHIND A BLUFF BODY, E.G. A FLAME-HOLDER (BEFORE CONFINEMENT INCREASES THE HOT-GAS VELOCITY).



  - FILM COOLING WHEN THE INJECTION VELOCITY IS LOWER THAN THAT OF THE SURROUNDING STREAM.
- UNSTEADY WAKES: • THAT BEHIND A DECELERATING BODY, E.G. AN INJECTED DROPLET.

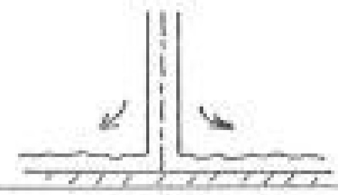
HTE 1 12	5/15	PRACTICAL RELEVANCE; EXAMPLES OF PLUMES.

- STEADY PLUMES: • SMOKE RISING ABOVE A FIRE WHEN THERE IS LITTLE WIND.
  - WARM WATER RISING FROM A LAKE-BOTTOM DISCHARGE POINT.
  - MOIST WARM AIR RISING ABOVE A COOLING TOWER.
  - DENSE CONCENTRATE FALLING DOWN THE CAVITY WALL IN SOLUTION MINING.

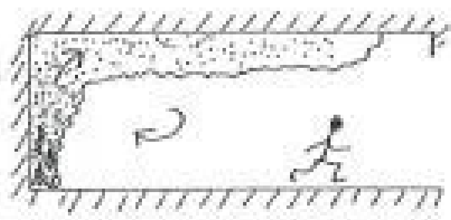
- UNSTEADY PLUMES: • THE STARTING OR STOPPING PROCESS OF ANY OF THE ABOVE.
  - THE AIR MOVEMENT INDUCED (AFTER BLAST-WAVE EFFECTS) BY A NUCLEAR-BOMB EXPLOSION.

HTE 1 12	6/15	PRACTICAL RELEVANCE; EXAMPLES OF LAYERS.

- STEADY LAYERS: • THE WARM-WATER LAYER RESULTING FROM SURFACE DISCHARGE OF THERMAL EFFLUENT INTO A RIVER.
  - THE FLOW OF WATER AWAY FROM THE POINT OF IMPINGEMENT OF A FALLING JET ON A PLATE.

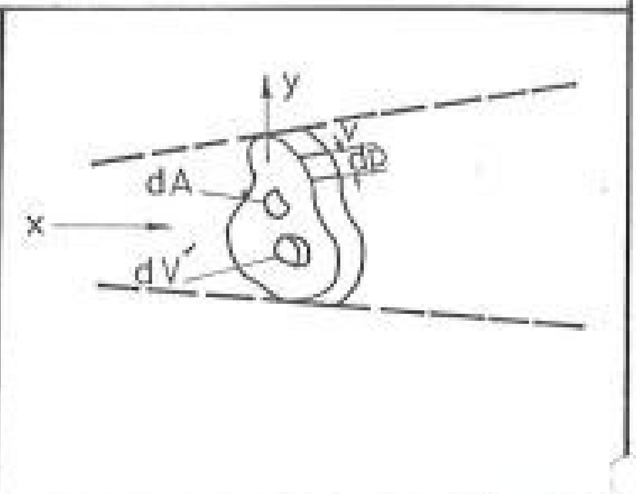


- UNSTEADY LAYERS: • SMOKE FROM A FIRE, SPREADING ALONG THE CEILING OF A CORRIDOR.
  - AN AVALANCHE OF SNOW.
  - A "TURBIDITY CURRENT", ON THE OCEAN FLOOR.



HTE 1 12	7/15	MATHEMATICAL DESCRIPTION OF JETS, ETC.; NOMENCLATURE.

- GEOMETRICAL:
  - V' ≡ VOLUME PER UNIT AXIAL DISTANCE, = A.
  - x ≡ AXIAL DISTANCE.
  - A ≡ AREA OF "SLICE" OF JET, ETC.
  - P ≡ PERIMETER OF SLICE.
  - y ≡ DISTANCE NORMAL TO PERIMETER.



- KINEMATIC:
  - u ≡ VELOCITY NORMAL TO FACE OF SLICE (POSITIVE WITH x).
  - v ≡ VELOCITY NORMAL TO PERIMETER SURFACE (POSITIVE OUTWARDS.

HTE 1 12	8/15	GENERAL CONSERVATION EQUATION APPLIED TO A "SLICE".

- EQUATION:

$$\frac{d}{dt}\int_{V'} \rho\phi dV' + \frac{d}{dx}\int_{A} \rho u\phi dA + \int_{P} \rho v\phi dP = \frac{d}{dx}\int \Gamma_\phi \frac{\partial\phi}{\partial x} dA + \int_{P} \Gamma_\phi \frac{\partial\phi}{\partial y} dP + \int_{V'} S_\phi dV'.$$

- NOTES:
  - THE CONTROL-VOLUME SHAPE MAY BE CHANGING WITH TIME, PROVIDED THAT u AND v ARE MEASURED RELATIVE TO THE MOVING SURFACES.
  - x CAN BE MEASURED ALONG JUST ONE CURVE THREADING THE SLICES; BUT THEN dP MAY NEED TO BE MULTIPLIED BY AN APPROPRIATE CURVATURE FACTOR.

HTE 1 12	9/15	CONSERVATION EQUATION FOR A SLICE, WITH NEGLIGIBLE BOUNDARY DIFFUSION.

- PRACTICAL RELEVANCE: • THE BOUNDARIES OF THE JET SUB-DOMAIN CAN USUALLY BE CHOSEN.
  - IT IS USEFUL TO SET THESE AT SUCH A DISTANCE THAT $\partial\phi/\partial y$ IS NEGLIGIBLE.
  - BECAUSE OF THE ELONGATED NATURE OF JETS, LAYERS, ETC., THE $\Gamma_\phi \frac{\partial\phi}{\partial x}$ TERMS ARE OFTEN NEGLIGIBLE COMPARED WITH THE $\rho u\phi$ TERMS.
- RESULTING EQUATION:

$$\frac{d}{dt}\int_{V'} \rho\phi dV' + \frac{d}{dx}\int_A \rho u\phi dA + \int_P \rho v\phi dP = \int_{V'} S_\phi dV'$$

- NOTE: WHEN A WALL ADJOINS THE SUB-DOMAIN, THE NORMAL GRADIENT WILL OFTEN NOT EQUAL ZERO.

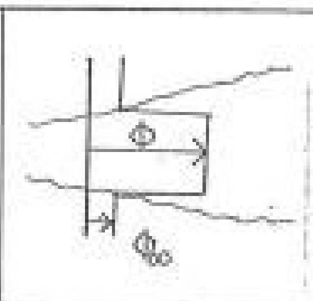
HTE 1 12	10/15	THE CONTINUITY EQUATION FOR A SLICE.

- DEFINING FEATURE: • PUT $\phi = 1$.
  - N.B. NEGLECT OF grad ϕ THEREFORE ENTAILS NO ERRORS.
- RESULTING EQUATION:

$$\frac{d}{dt}\int_{V'} \rho dV' + \frac{d}{dx}\int_A \rho u dA + \int_P \rho v dP = \int_{V'} S_{mass} dV'$$

- DISCUSSION: • FOR STEADY-STATE, NO-SOURCE FLOWS, WHICH ARE COMMON: $\frac{d}{dx}\int_A \rho u dA + \int_P \rho v dP = 0$.
  - EVEN IN TRANSIENT PROCESSES THIS EQUATION HOLDS PROVIDED ρ IS INVARIANT.
  - THERE IS ALSO A BULK-CONTINUITY FORM.

HTE 1 12	11 15	THE "TOP-HAT" PROFILE; DEFINITION AND RESULTING EQUATIONS.



- DEFINITION: • ϕ IS SUPPOSED TO BE UNIFORM IN THE JET REGION, AND EQUAL TO ϕ_∞ OUTSIDE IT.
  - AXIAL ϕ GRADIENTS ARE NEGLECTED.
  - S_ϕ IS ALSO UNIFORM.
  - $v < 0$.
- RESULTING EQUATION:

$$\frac{d}{dt}\left(\phi\int_{V'}\rho dV'\right) + \frac{d}{dx}\left(\phi\int_{A}\rho u dA\right) + \phi_\infty\int_{P}\rho v dP = S_\phi V.$$

- COMBINATION WITH THE CONTINUITY EQUATION, WITH ϕ_∞ A CONSTANT, LEADS TO:

$$\left(\int_{V'}\rho dV'\right)\frac{d}{dt}(\phi - \phi_\infty) + \frac{d}{dx}\left((\phi - \phi_\infty)\int_{A}\rho u dA\right) = S_\phi V.$$

HTE 1 12	12 15	THE "TOP-HAT" PROFILE; DISCUSSION.

- IF ρu AND ρ ARE TAKEN AS UNIFORM OVER THE SLICE:

$$\rho V' \frac{d}{dt}(\phi - \phi_\infty) + \frac{d}{dx}\{(\phi - \phi_\infty)\rho u A\} = S_\phi V.$$

- OTHER FORMS OF EQUATION CAN BE DERIVED FOR THE CASE IN WHICH ϕ_∞ VARIES WITH t AND x.
- THE STEADY-STATE FORM IS: $\frac{d}{dx}\{(\phi - \phi_\infty)\rho u A\} = S_\phi V.$
- IN THE ABSENCE OF SOURCES: $\frac{d}{dx}\{(\phi - \phi_\infty)\rho u A\} = 0.$
- THIS IS RELEVANT TO WARM-WATER MIXING, TO ROCKET EXHAUSTS, ETC.

HTE 1 12	13/15	ONE-PARAMETER PROFILES; DESCRIPTION.

- RESTRICTIONS (FOR SIMPLICITY, NOT FROM NECESSITY):
  - ZERO BOUNDARY GRADIENTS.
  - UNIFORM DENSITY.
  - STEADY STATE.
  - ZERO SOURCES.
- ASSUMPTIONS:
  - $\phi \equiv \phi_\infty(x) + \Phi(x).\tilde{\phi}$
  - $u \equiv u_\infty(x) + U(x).\tilde{u}$
  - $\tilde{\phi}$ AND $\tilde{u}$ ARE SELF-SIMILAR FUNCTIONS OF POSITION.
- RESULTING CONSERVATION EQUATION:

$$\frac{d}{dx}\{(\phi_\infty\int_A \tilde{u}\,d\tilde{A} + \Phi\int_A \tilde{u}\tilde{\phi}\,d\tilde{A})UA\} + \phi_\infty\int_P v\,dP = 0.$$

- ASSOCIATED MASS-CONSERVATION EQUATION:

$$\frac{d}{dx}\{UA\int_A \tilde{u}\,d\tilde{A}\} + \int_P v\,dP = 0.$$

HTE 1 12	14/15	ONE-PARAMETER PROFILES: THE EQUATION FOR Φ.

- MANIPULATION: MULTIPLY MASS-CONSERVATION EQUATION BY ϕ_∞ AND SUBTRACT FROM ϕ CONSERVATION.
- RESULT:

$$UA\int_A \tilde{u}\,d\tilde{A}.\frac{d\phi_\infty}{dx} + \frac{d}{dx}\{\Phi A\int_A \tilde{u}\tilde{\phi}\,dA\} = 0$$

- UTILITY:
  - USUALLY $\frac{d\phi_\infty}{dx}$ IS SPECIFIED.
  - THE PREDICTION OF THE FLOW HAS BEEN REDUCED TO THAT OF INTEGRATING FIRST-ORDER DIFFERENTIAL EQUATIONS.
  - THIS IS ALWAYS EASY BY NUMERICAL MEANS.

HTE 1 12	15/15	FINAL REMARKS CONCERNING JETS, WAKES, LAYERS, ETC.

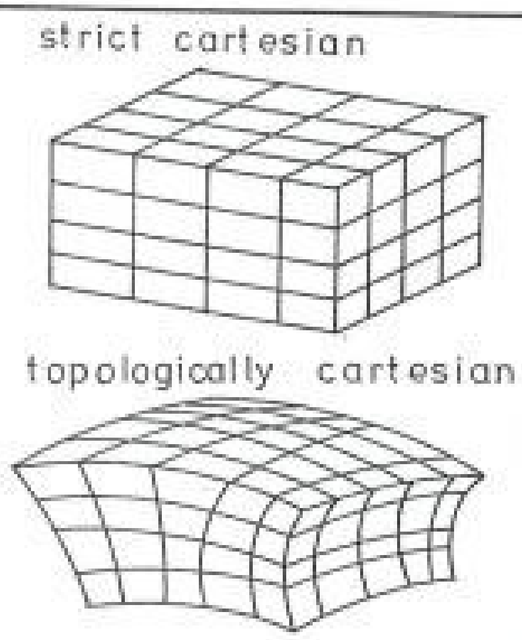
- FROM PANEL 9 ONWARDS, THE EQUATIONS WERE ILLUSTRATIVE RATHER THAN COMPREHENSIVE. THE MORE GENERAL EQUATIONS TAKE MORE SPACE, BUT ARE EASY TO WRITE.
- FOR MANY PRACTICAL PURPOSES THE JET ~ ETC. SUB-DOMAIN IS ADEQUATE; CALCULATIONS ARE CHEAPLY MADE; AND EXPERIENCE PERMITS CHOICES OF PROFILE THAT FAVOUR ACCURACY.
- EVEN WHEN FINITE-DIFFERENCE ANALYSES ARE TO BE MADE FOR THE GREATER PART OF THE FLOW, REGIONS OF STEEP GRADIENT ARE OFTEN BEST ANALYSED AS JETS, ETC.

HTE 1 13	$\frac{1}{15}$	LECTURE 13. FINITE-DIFFERENCE GRIDS (CELLULAR SUB-DIVISION OF SPACE).

- CONTENTS:
  - THE TOPOLOGICALLY-CARTESIAN CELL SYSTEM.
  - TWO-DIMENSIONAL EXAMPLES.
  - THREE-DIMENSIONAL EXAMPLES.
  - REPRESENTATION OF:
    - TRANSIENT,
    - CONVECTIVE ,
    - DIFFUSIVE,
    - SOURCE TERMS.
  - COMBINATION OF CONVECTIVE AND DIFFUSIVE TERMS.

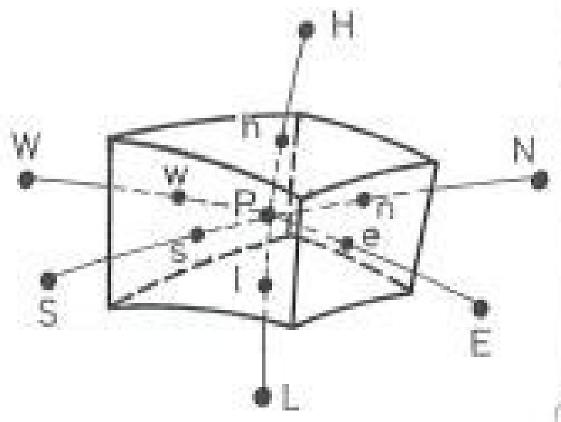
HTE 1 13	$\frac{2}{15}$	THE TOPOLOGICALLY-CARTESIAN CELL SYSTEM; THE BASIC IDEA.

- THE WHOLE SPACE IS FILLED BY NON-OVERLAPPING CELLS.
- EACH CELL HAS 6 SIDES, 12 EDGES AND 8 VERTICES.
- EACH CENTRAL CELL HAS 6 NEIGHBOURS; EACH SURFACE CELL HAS 5; EACH EDGE CELL HAS 4; EACH CORNER CELL HAS 3.
- CELLS CAN BE DEFINED BY 3 SETS OF NON-INTERSECTING SURFACES, WHICH MAY CUT ORTHOGONALLY BUT NEED NOT.



HTE 1 13	3/15	THE TOPOLOGICALLY-CARTESIAN CELL SYSTEM; GRID-POINT VALUES.

- CELL-CENTRE POINTS (P) ARE DEFINED WITHIN EACH CELL.
- ITS NEIGHBOURS ARE REFERRED TO AS POINTS N, S, E, W, H, L.
- LINES JOINING P TO ITS NEIGHBOURS CUT CELL FACES AT n, s, e, w, h, l.



- ϕ VALUES "AT" P, N, S, ... ETC. ARE REGARDED AS REPRESENTING ϕ'S WITHIN THE WHOLE CELLS.
- ϕ VALUES (AND ϕ GRADIENTS) "AT" n, s, e, ... ARE REGARDED AS REPRESENTING VALUES OVER THE WHOLE OF THE FACES.

HTE 1 13	4/15	THE TOPOLOGICALLY-CARTESIAN CELL; APPLICATION OF THE GENERAL CONSERVATION EQUATION.

- THE EQUATION:

$$\frac{\partial}{\partial t}(\rho\phi V)_P + \sum_{\substack{n,s,e,w\\h,l}} (\phi\rho\vec{u} - \Gamma_\phi \text{ grad } \phi).\vec{A} = (S_\phi V)_P$$

- NOTES: • V MAY VARY WITH TIME; BUT IN THAT CASE $\vec{u}$ MUST BE MEASURED RELATIVE TO THE MOVING SURFACE.
  - $\sum_{n,s,e...}$ MEANS THE SUM OF 6 TERMS, ONE FOR EACH CELL FACE.
  - ϕ AND S_ϕ HAVE BEEN ASSUMED UNIFORM OVER THE CELL IN THE TRANSIENT AND SOURCE TERMS RESPECTIVELY.

HTE 1 13	5/15	THE TOPOLOGICALLY-CARTESIAN CELL; RELATION TO "FINITE-DIFFERENCE" EQUATIONS.

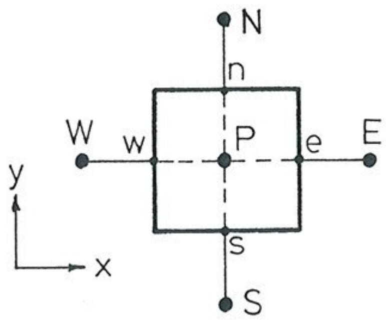
- THE FDE: ϕ'S AT NEIGHBOURING POINTS ARE TO BE CONNECTED BY "LINEAR" EQUATIONS OF FORM (SEE LECTURE 14):

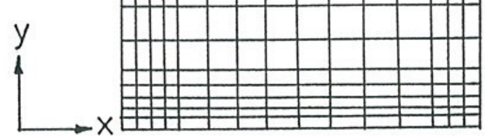
$$\phi_P = a_n\phi_N + a_s\phi_S + a_e\phi_E + a_w\phi_W + a_h\phi_H + a_l\phi_L + a_p\phi_{P-} + b$$

- NOTES: • THE FDE IS TO BE OBTAINED FROM THE CONSERVATION EQUATION BY REPRESENTATION OF ITS TERMS THROUGH ϕ_N, ϕ_S, ... a, c_{p-}, c_w, ... , c_p, b.
  - THIS REPRESENTATION NECESSITATES FURTHER ASSUMPTIONS, WHICH CAN BE MADE IN MORE THAN ONE WAY.
  - ϕ_{P-} IS THE VALUE OF ϕ_P AT AN EARLIER INSTANT OF TIME.
  - a REPRESENTS THE (CONSTANT PART OF THE) SOURCE TERM.
  - "LINEAR" IS IN QUOTES BECAUSE a, c's AND b DEPEND ON ϕ'S.

HTE 1 13	6/15	EXAMPLES OF 2D CELL SYSTEMS: PLANE, STRICT CARTESIAN.

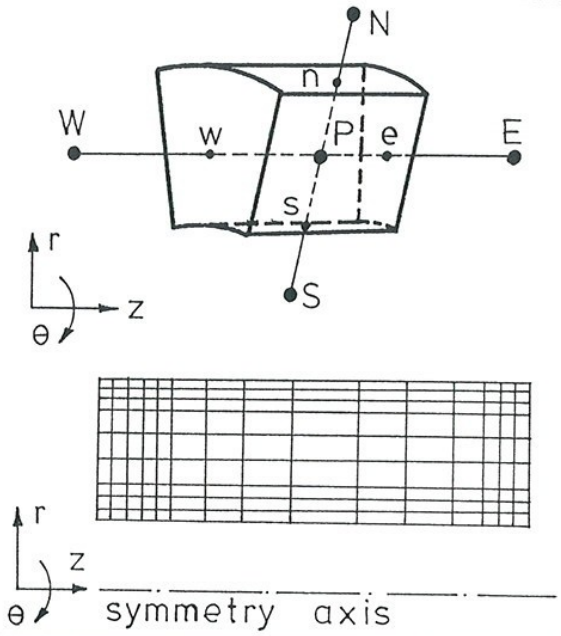
- ϕ'S VARY ONLY WITH x AND y.
- CELLS ARE OF UNIFORM DEPTH (E.G. UNITY) IN 3RD (z) DIRECTION.
- CELLS CAN BE NON-UNIFORMLY DIMENSIONED WITHIN THE LIMITS OF RECTANGULARITY. SEE →
- THE FDE LACKS H ("high") and L ("low") terms





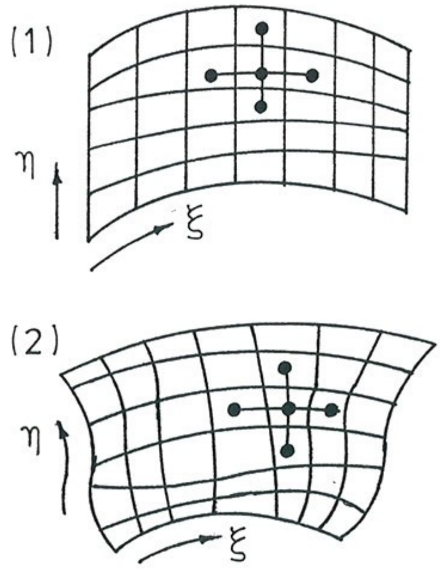
HTE 1 13	7 15	EXAMPLES OF 2D CELL SYSTEMS: POLAR COORDINATE; STRICT CARTESIA

- ϕ'S VARY ONLY WITH z AND r (NOT θ).
- CELL DEPTH INCREASES WITH r.
- CELL SHAPES IN THE $r \sim z$ PLANE ARE LIMITED IN THE SAME WAY AS THOSE IN $x \sim y$ PLANE.
- FDE IS AS FOR PANEL 6; BUT r ENTERS THE a'S AND b.



HTE 1 13	8 15	EXAMPLES OF 2D CELL SYSTEMS; TOPOLOGICALLY CARTESIAN.

- IN ①, ONE SET OF LINES IS PARALLEL, THE OTHER CURVED. THE LATTER MAY BE STREAMLINES.
- IN ②, BOTH ARE CURVED; AND THE TWO SETS MAY BE ORTHOGONAL.
- BOTH PLANE AND AXI-SYMMETRICAL SYSTEMS MAY BE SUBDIVIDED IN THIS WAY.
- STREAMLINES ARE CONVENIENT; CONVECTION IS ZERO ACROSS THEM.

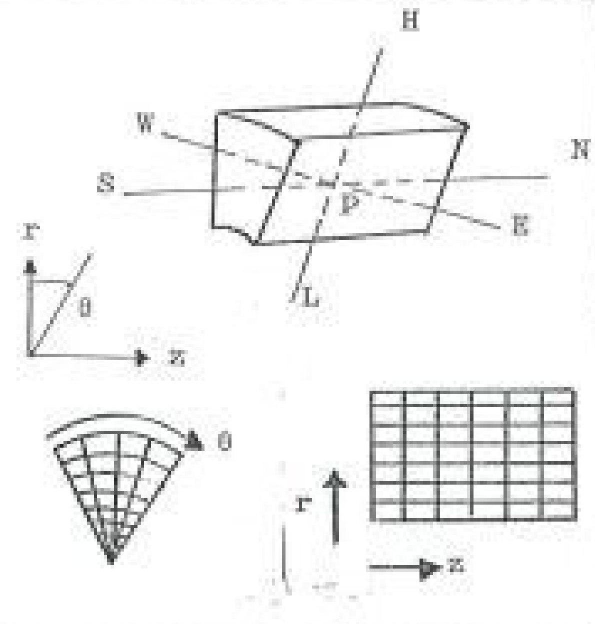


HTE 1 13	9 15	EXAMPLES OF 2D CELL SYSTEMS; THE DISTINCTION BETWEEN FIXED AND FLOATING GRIDS.

- FIXED GRIDS:
  - THE LOCATIONS OF CELL BOUNDARIES AND CENTRE POINTS ARE SPECIFIED AT THE START OF COMPUTATION AND REMAIN UNCHANGED.
  - IN GENERAL, THE MASS FLOW RATES ACROSS CELL WALLS ARE NOT KNOWN BEFOREHAND, AND MUST BE COMPUTED.
- FLOATING GRIDS:
  - THE LOCATIONS OF AT LEAST SOME CELL BOUNDARIES ARE FIXED IMPLICITLY RATHER THAN EXPLICITLY, AND VARY AS THE COMPUTATION PROCEEDS.
  - OFTEN THE IMPLICIT SPECIFICATION IS SUCH AS TO DEFINE THE MASS FLOW RATES; E.G. THESE ARE ZERO FOR STREAM-LINE CELL WALLS.

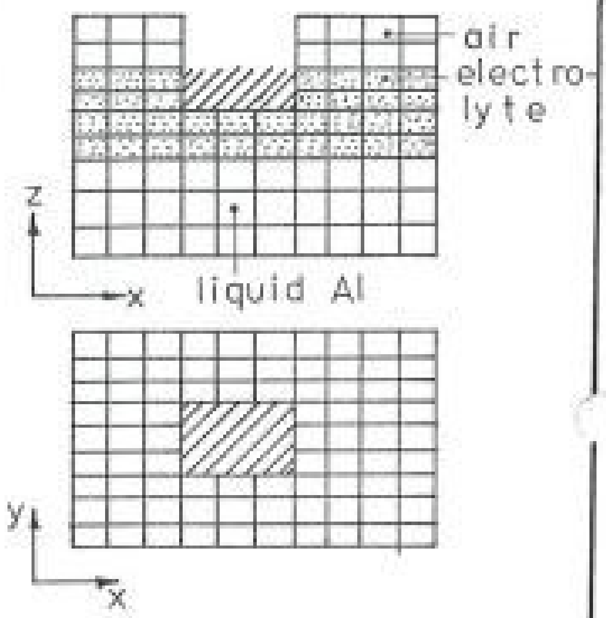
HTE 1 13	10 15	EXAMPLES OF 3D CELL SYSTEMS; POLAR COORDINATE, STRICT CARTESIAN.

- ϕ'S VARY WITH r, z AND θ.
- CELLS ARE SHAPED AS FOR PANEL 7.
- IN $r \sim \theta$ PLANE, CELL WALLS ARE RADII AND CONCENTRIC CIRCLES.
- IN $r \sim z$ PLANE THEY ARE AS FOR PANEL 7.
- THE FDE IS AS IN PANEL 5.



HTE 1 13	11/15	EXAMPLES OF 3-D CELL SYSTEMS; A GRID WHICH FLOATS IN ONE DIRECTION ONLY.

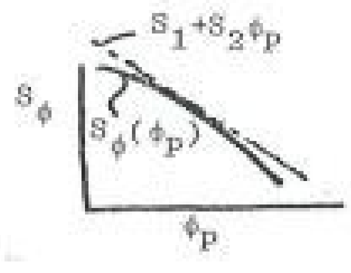
- PRACTICAL EXAMPLE: THE ALUMINIUM SMELTER.
- DEFINING CONDITIONS:
  - THE AIR ~ ELECTROLYTE, AND ELECTROLYTE-Al INTERFACES COINCIDE WITH CELL BOUNDARIES.
  - WITHIN EACH FLUID VERTICAL SUB-DIVISIONS ARE UNIFORM FOR EACH x ~ y.
- OTHER EXAMPLES: ENVIRONMENTAL FLOWS.



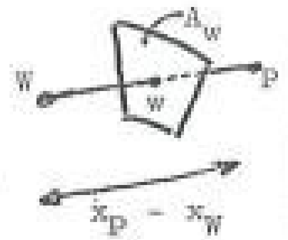
HTE 1 13	12/15	REPRESENTATION OF TERMS IN THE CONSERVATION EQUATION; TRANSIENT, AND SOURCE.

- TRANSIENT TERMS: $\frac{\partial}{\partial t}(\rho\phi V)_P = \frac{(\rho\phi V)_P - (\rho\phi V)_{P-}}{\delta t}$

  NOTE: SUBSCRIPT P NOW REFERS TO TIME t, AND SUBSCRIPT P- TO TIME $t - \delta t$.
- SOURCE TERMS: • $(S_\phi V)_P = (S_1 + S_2\,\phi_P)V_P$, WHERE S_1 AND S_2 ARE CHOSEN SO THAT THE LINE TOUCHES THE CURVE AT A GUESSED VALUE OF ϕ_P.
  - THIS IS THE "LINEARISED-SOURCE" TREATMENT.
  - SEE PANELS 14.7,8 FOR FURTHER DISCUSSION.



HTE 1 13	13/15	REPRESENTATION OF TERMS IN THE CONSERVATION EQUATION; DIFFUSION AND CONVECTION.



- CONVECTION: "CENTRAL-DIFFERENCE" REPRESENTATION IS:
  $(\phi\rho\vec{u})_w = \phi_w\,(\rho\vec{u})_w,$

  $\phi_w = \frac{1}{2}(\phi_W + \phi_P).$

  (NOTE: $(\rho\vec{u})_w$ IS PROBABLY IN STORE; SEE LECTURE 15.)
- DIFFUSION: $(\Gamma_\phi \text{ grad } \phi)_w = \Gamma_{\phi,w}\,\dfrac{\phi_P - \phi_W}{x_P - x_W}$

  $$\Gamma_{\phi,w} = \tfrac{1}{2}(\Gamma_{\phi,W} + \Gamma_{\phi,P})$$
- NOTE: IF THE LINE WP IS NOT NORMAL TO AREA A_w AN APPROPRIATE COSINE MUST BE INTRODUCED.

HTE 1 13	14/15	REPRESENTATION OF TERMS; INTERACTION OF DIFFUSION AND CONVECTION.

- THE PANEL 13 REPRESENTATION IS VALID ONLY WHEN $\left|(\rho u)_w\,(x_P - x_W)/\Gamma_{\phi,w}\right| \ll 1.$
- OTHERWISE ϕ_w TENDS TO ϕ_W IF $u_w > 0$, AND TO ϕ_P WHEN $u_w < 0$, AS SOLUTION OF THE DIFFERENTIAL EQUATION WILL SHOW.
- IN THE "UPWIND-DIFFERENCE SCHEME", DIFFUSION IS COMPUTED AS IN PANEL 13, BUT ϕ_w IS TAKEN AS ϕ_W FOR $u_w > 0$ AND ϕ_P FOR $u_w < 0$.
- IN GENERAL IT IS BEST TO REGARD DIFFUSION AND CONVECTION AS INTERACTING: A SINGLE "FINITE-DIFFERENCE" EXPRESSION IS NEEDED FOR THEIR COMBINED EFFECT.
- SEE LECTURE 14 FOR FURTHER DISCUSSION.

HTE 1 13	15 15	FINAL REMARKS

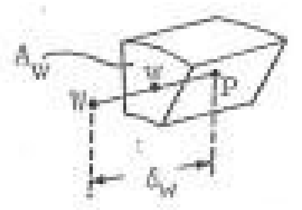
- CELLULAR SUB-DIVISION OF SPACE IS A FLEXIBLE AID TO ANALYSIS OF THE VARIOUS TRANSPORT PROCESSES: THE SHAPES AND DISTRIBUTIONS OF CELLS CAN BE CHOSEN TO SUIT THE PROBLEM.
- THE TOPOLOGICALLY-CARTESIAN SYSTEM IS CONVENIENT, BECAUSE IT PERMITS CELLS TO BE REFERRED TO IN AN ORDERLY WAY (E.G. BY THREE INDICES).
- OTHER KINDS OF SUB-DIVISION ARE POSSIBLE, E.G. TRIANGULAR, TETRAHEDRAL (FINITE-ELEMENT).
- MORE MUST BE SAID ABOUT THE DERIVATION OF THE FDE'S FROM THE CONSERVATION EQUATIONS; THERE IS NOT ONE SINGLE CORRECT WAY.

HTE 1 14	1 15	LECTURE 14 FINITE-DIFFERENCE EQUATIONS

- CONTENTS:
  - FLUXES ACROSS CELL BOUNDARIES.
  - VOLUME TERMS.
  - THE CHOICE OF TIME LEVEL.
  - THE FDE FOR A TOPOLOGICALLY-CARTESIAN CELL.
- NOTE:
  - DISCUSSION WILL BE IN TERMS OF THE GENERAL VARIABLE ϕ, STANDING AS BEFORE FOR MOMENTUM PER UNIT MASS, STAGNATION ENTHALPY, MASS FRACTION, ETC.

HTE 1 14	2 15	FLUXES ACROSS CELL BOUNDARIES: DIFFUSION AND CONVECTION.

- EXAMPLE CONSIDERED:
  THE WEST FACE OF A TOPOLOGICALLY CARTESIAN CELL, HAVING AREA A_w.



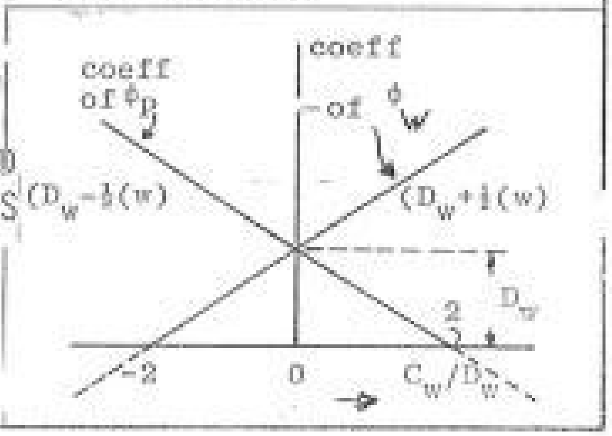
- FD EXPRESSION FOR THE DIFFUSION FLUX: $A_w J_{\phi,w} = -A_w \Gamma_{\phi,w} (\phi_P - \phi_W)/\delta_w$.
- FD EXPRESSION FOR THE CONVECTIVE FLUX (A_w OMITTED):
  FLUX = $(\rho u)_w \phi_w$, WHERE:

EITHER: $\phi_w = \frac{1}{2}(\phi_W + \phi_P)$ IN "CENTRAL-DIFFERENCE SCHEME",

OR : $\phi_w = \phi_W$ FOR $(\rho u)_w > 0$, ϕ_P FOR $(\rho u)_w < 0$ } IN "UPWIND-DIFFERENCE SCHEME".

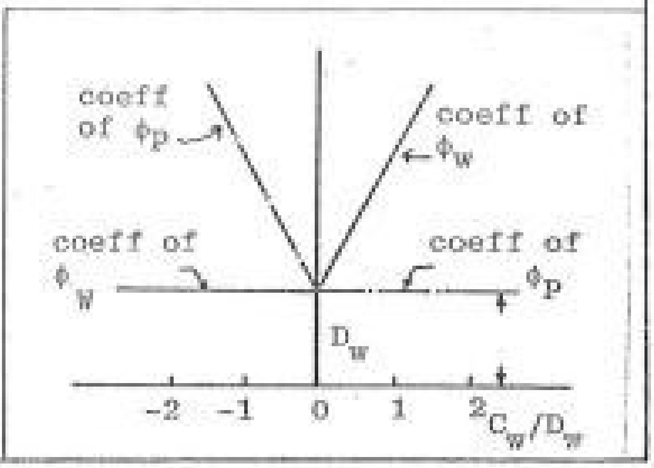
HTE 1	3	TOTAL-FLUX EXPRESSIONS;
14	15	CENTRAL DIFFERENCE. $(\phi_w = \frac{1}{2}(\phi_W + \phi_P))$

- NOTATION: $C_w \equiv (\rho u A)_w$; $D_w \equiv \Gamma_{\phi,w} A_w/\delta_w$;
  $A_w J_{tot,\phi,w} \equiv C_w \phi_w + D_w(\phi_W - \phi_P)$.
- CENTRAL-DIFFERENCE EXPRESSION:
  $A_w J_{tot,\phi,w} = (D_w + \frac{1}{2} C_w)\phi_W - (D_w - \frac{1}{2} C_w)\phi_P$.
- DISCUSSION: • NEGATIVE COEFFICIENTS IMPLY $J_{tot,\phi}$ DECREASES AS ϕ_W INCREASES; AND IT INCREASES AS ϕ_P INCREASES.
- THE RESULT FOLLOWS FROM NEGLECTING INFLUENCE OF u_w ON ϕ_w (SEE PANEL 13.14)
- THIS IS UNREALISTIC.



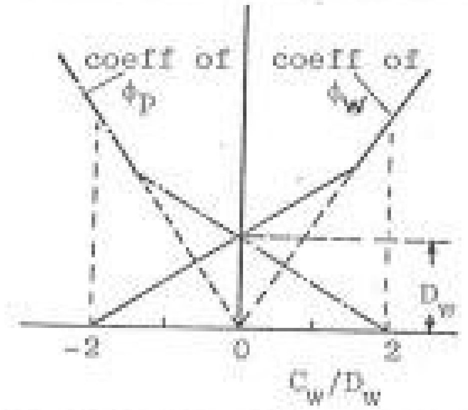
HTE 1	4	TOTAL-FLUX EXPRESSIONS;
1.4	15	UPWIND (ALSO CALLED DONOR-CELL).

- UPWIND-DIFFERENCE EXPRESSIONS FOR $J_{tot,\phi,w}$:
  FOR $C_w \geq 0$: $A_w J_{tot,\phi,w} = (D_w + C_w)\phi_W - D_w\phi_P$;
  FOR $C_w < 0$: $A_w J_{tot,\phi,w} = D_w\phi_W - (D_w - C_w)\phi_P$.
- DISCUSSION:
  - NO NEGATIVE COEFFICIENTS NOW APPEAR.
  - EVEN A VERY HIGH $(-C_w)$ PERMITS SOME INFLUENCE OF ϕ_W TO AFFECT THE CELL SURROUNDING P.



HTE 1 14	5/15	TOTAL-FLUX EXPRESSIONS; HYBRID.

- FOR $-2 < C_w/D_w < +2$: AS FOR CENTRAL DIFFERENCES.
- FOR $2 \leq C_w/D_w$: $A_w J_{tot,\phi,w} = C_w \phi_W$.
- FOR $-2 \geq C_w/D_w$: $A_w J_{tot,\phi,w} = -C_w \phi_P$.
- DISCUSSION:
  - COEFFICIENTS FALL TO ZERO, BUT DO NOT BECOME NEGATIVE.
  - THIS FITS REALITY FAIRLY WELL, ALTHOUGH DIFFUSIVE INFLUENCE IS NEVER ENTIRELY ZERO IN PRACTICE.

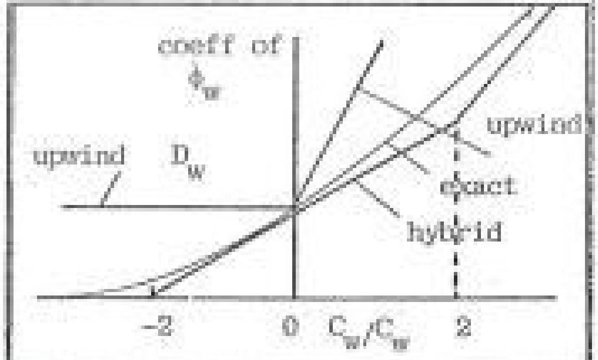


HTE 1 14	6/15	TOTAL-FLUX EXPRESSIONS: DISCUSSION.

- EXACT SOLUTION OF THE 1D DIFFERENTIAL EQUATION WITH UNIFORM Γ: AND WITHOUT SOURCES GIVES:

$$A_w J_{tot,\phi,w} = \frac{C_w}{1-\exp\left(\frac{-C_w}{D_w}\right)} \cdot \phi_W - \frac{C_w \exp\left(-\frac{C_w}{D_w}\right) \phi_P}{1-\exp\left(-\frac{C_w}{D_w}\right)}.$$

- NOTES: • C_w/D_w HAS THE FORM OF A PECLET NUMBER.
  - THE HYBRID FORMULA CORRESPONDS TO THE EXACT WELL AT $C_w/D_w = 0, -\infty$ AND $+\infty$.
- IT IS RECOMMENDED FOR MANY PURPOSES.



HTE 1 14	7 15	REPRESENTATION OF VOLUME TERMS; TRANSIENT TERM.

- THE EXPRESSION:

$$\frac{\partial}{\partial t}\int_V \phi\rho dV = \frac{(\phi\rho V)_P - (\phi\rho V)_{P-}}{\delta t}$$

WHERE P AND P- REFER TO THE TIMES t AND $t-\delta t$, AND δt IS A FINITE TIME INCREMENT.

- NOTES: • WHEN THE GRID IS FIXED, THE TERM BECOMES $[(\phi\rho)_P - (\phi\rho)_{P-}]V/\delta t$.
  - ϕ AND ρ ARE TAKEN AS UNIFORM WITHIN THE CELL.
  - OTHER PRESUMPTIONS ARE POSSIBLE.
  - $(\rho V)_P/\delta t$ AND $(\rho V)_{P-}/\delta t$ CAN BE USEFULLY THOUGHT OF AS OUTFLOW AND INFLOW RATES THROUGH TIME BOUNDARIES, ANALOGOUS TO $(\rho uA)_w$ ETC.

HTE 1 14	8 15	REPRESENTATION OF VOLUME TERMS; THE SOURCE TERM.

- USUAL PRESUMPTION: S_ϕ IS UNIFORM THROUGHOUT THE CELL.
- CONSEQUENCE: SOURCE TERM IS $S_{\phi,P}V$.
- QUESTIONS: • IF S_ϕ IS $S_\phi(\phi)$, SHOULD ϕ_P OR ϕ_{P-} BE USED?
  - FOR THAT MATTER, WHICH SHOULD WE HAVE BEEN CONSIDERING IN THE DIFFUSION AND CONDUCTION TERMS?
  - IF S_ϕ CAN BE EXPRESSED AS $S_1 + S_2\,\phi$ (PANEL 13.12), SHOULD WE USE

    $S_1 + S_2\,\phi_{P-}$, $S_1 + S_2\,\phi_P$, OR $S_1 + S_2\,\frac{1}{2}(\phi_P + \phi_{P-})$?
- RECOMMENDED ANSWERS WILL BE GIVEN LATER.

HTE 1 14	9/15	CHOICE OF TIME LEVEL; ALTERNATIVES.

- IN THE TIME-DEPENDENT TERM, THERE IS NO CHOICE; THE DIFFERENCE BETWEEN $(\rho\phi V)_P$ AND $(\rho\phi V)_{P-}$ MUST APPEAR.
- IN THE DIFFUSION, CONVECTION AND COMBINED-FLUX TERMS, ONE COULD:
  - CHOOSE ALL ϕ'S FOR EARLIER TIME (ϕ_{P-}, ϕ_{W-}, ...).
  - CHOOSE ALL NEIGHBOUR ϕ'S FOR EARLIER TIME (ϕ_{W-}, ϕ_{S-}, ...) BUT ϕ_P FOR LATER TIME.
  - HAVE DIFFERENT PRACTICES FOR THE CONVECTION AND DIFFUSION COMPONENTS.
  - USE WEIGHTED MEANS OF EARLIER OR LATER VALUES.
  - USE ESTIMATES OF ϕ'S AT INTERMEDIATE TIMES.

HTE 1 14	10/15	CHOICE OF TIME INTERVAL; THE EXPLICIT SCHEME.

- DEFINITION: USE ϕ_{P-}, ϕ_{W-}, ETC., FOR DIFFUSION, CONVECTION AND SOURCES; BUT ϕ_P APPEARS IN THE TIME-DEPENDENT TERM.
- ADVANTAGE:
  - FDE CAN BE REPRESENTED EXPLICITLY IN TERMS OF EARLIER-TIME VALUES, WHICH MAY BE SUPPOSED KNOWN.
  - THE SOLUTION PROCEDURE IS THEREFORE EASY.
- DISADVANTAGE:
  - INSTABILITY (ERROR AMPLIFICATION) OCCURS WHEN δt IS LARGE.
- NOTES:
  - THIS REQUIRES MATHEMATICAL PROOF. (SEE PANEL 18.7)
  - THE SCHMIDT METHOD FOR HEAT CONDUCTION IS OF THIS KIND.

HTE 1 14	11/15	CHOICE OF TIME INTERVAL; THE CRANK-NICHOLSON SCHEME.

- DEFINITION: $\frac{1}{2}(\phi_P + \phi_{P-})$, $\frac{1}{2}(\phi_W + \phi_{W-})$, ETC., APPEAR IN ALL DIFFUSION, CONVECTION AND SOURCE TERMS.
- ADVANTAGE: • ARBITRARILY LARGE δt'S CAN BE CHOSEN WITHOUT INSTABILITY.
  - THEREFORE THE COMPUTER TIME FOR A GIVEN UNSTEADY PROCESS MAY BE REDUCED, ESPECIALLY WHEN THE FINAL STEADY STATE IS OF MAJOR INTEREST.
- DISADVANTAGE: • EQUATIONS ARE NOW SIMULTANEOUS, REQUIRING MATRIX INVERSION, OR ITERATION.
- NOTE: THIS SCHEME IS POPULAR.

HTE 1 14	12/15	CHOICE OF TIME INTERVAL; THE FULLY-IMPLICIT SCHEME.

- DEFINITION: ALL ϕ'S, EXCEPT THE ϕ_{P-} IN THE TRANSIENT TERM, ARE CHOSEN FOR THE END OF THE TIME INTERVAL (I.E. ϕ_P, ϕ_W, ETC.).
- ADVANTAGE: • GREATER ACCURACY THAN CRANK-NICHOLSON FOR VERY LARGE δt (BUT LESS ACCURACY FOR SMALLER δt).
  - GREATER ALGEBRAIC SIMPLICITY (ONE TERM INSTEAD OF TWO).
  - GREATER SUITABILITY FOR STEADY-STATE CALCULATIONS (δt IS PUT TO ∞, SO THE TRANSIENT TERM DISAPPEARS).
- NOTE: THIS IS EMPLOYED IN ALL OUR METHODS; BUT THEY COULD ALSO EMPLOY THE CRANK-NICHOLSON SCHEME, FOR EXAMPLE.

HTE 1	13	FDE FOR A TOPOLOGICALLY-CARTESIAN CELL;
14	15	DEFINITIONS.

- LET $c_W \equiv$ COEFFICIENT OF ϕ_W IN $A_w J_{tot,\phi,w}$,
  $c_E \equiv$ COEFFICIENT OF ϕ_E IN $A_e J_{tot,\phi,e}$, ETC.
- THEN COEFFICIENTS OF ϕ_P IN THESE EXPRESSIONS ARE
  $c_W - C_w$, $c_E - C_e$, ETC. $(C_e \equiv -A_e \rho_e u_e)$
- LET $S_\phi V \equiv a + b\phi_P$.
- LET $(\rho V)_{P-}/\delta t \equiv c_{P-}$.
- LET $c_P \equiv c_{P-} + c_W + c_E + c_N + c_S + c_H + c_L$.

HTE 1	14	FDE FOR A TOPOLOGICALLY-CARTESIAN CELL;
14	15	STATEMENT.

- SUBSTITUTION IN THE PANEL 13 - 4 EQUATION YIELDS:

$$\frac{(\phi\rho V)_P}{\delta t} - \phi_{P-}c_{P-} = c_W\phi_W + c_E\phi_E + c_S\phi_S + c_N\phi_N + c_H\phi_H + c_L\phi_L$$
$$- \{(c_W + c_E + c_S + c_N + c_H + c_L)$$
$$- (C_w + C_e + C_s + C_n + C_h + C_e)\}\ \phi_P + a + b\ \phi_P.$$

- FROM CONTINUITY: $\frac{(\rho V)_P}{\delta t} - C_{P-} = C_w + C_e + C_s + C_n + C_h + C_l$.
- CONSEQUENTLY:

$$\boxed{\phi_P = \frac{a + c_{P-}\phi_{P-} + c_W\phi_W + c_E\phi_E + c_S\phi_S + c_N\phi_N + c_H\phi_H + c_L\phi_L}{c_P - b}}$$

- NOTE: APART FROM THE a AND b TERMS, ϕ_P IS A WEIGHTED AVERAGE OF ϕ_{P-} AND THE NEIGHBOUR VALUES.

HTE 1 14	15/15	FINAL REMARKS

- THE PANEL 14 EQUATION REPRESENTS THE GENERAL FORM.
- ALL THE c'S ARE NECESSARILY POSITIVE, BY REASON OF THEIR DEFINITIONS.
- THE FORM IS THE SAME AS THAT OF THE PANEL 13 - 5 EQUATION, AS IT SHOULD BE.
- SOME OF THE c'S ARE ZERO, OR VERY SMALL, WHEN THE APPROPRIATE PECLET NUMBERS ARE LARGE. THIS PROVES TO BE IMPORTANT LATER.
- WE MUST SHORTLY CONSIDER HOW THE EQUATIONS ARE TO BE SOLVED.

HTE 1 15	1/15	LECTURE 15. THE "STAGGERED" FINITE-DIFFERENCE GRID.

- CONTENTS:
  - THE PROBLEM OF INTERPOLATION.
  - THE "STAGGERED" GRID.
  - THE CONTINUITY EQUATION FOR A CELL.

- NOTE: THE "STAGGERED" GRID IS AN ARRANGEMENT OF GRID NODES, FOR VELOCITIES DISTINGUISHED FROM OTHER VARIABLES, INTRODUCED BY HARLOW AND OTHERS AT LOS ALAMOS AND EMPLOYED IN METHODS DEVELOPED IN THE HEAT TRANSFER SECTION AT IMPERIAL COLLEGE.

HTE 1 15	2/15	THE PROBLEM OF INTERPOLATION

- IF Γ'S, $\vec{u}$'S, ρ'S, ϕ'S, ETC., ARE KNOWN ONLY FOR GRID POINTS (P, N, S, E,...), BUT ARE NEEDED, AS HAS BEEN SEEN, AT CELL WALLS, INTERPOLATION FORMULAE ARE NEEDED.
- EXAMPLES HAVE APPEARED IN LECTURE 13, VIZ.:

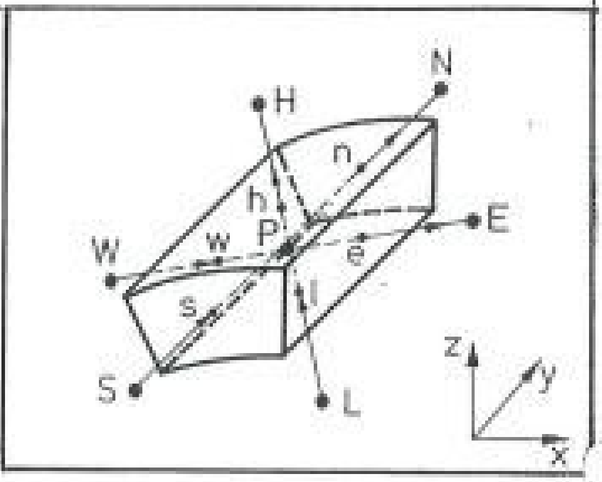
  $\Gamma_{\phi,w} = \frac{1}{2}(\Gamma_{\phi,W} + \Gamma_{\phi,P})$,

  $\phi_w = \alpha\,\phi_W + (1-\alpha)\,\phi_P$ WITH $\alpha = \frac{1}{2}$, 0 OR 1.

- THE QUESTION ARISES: WOULD IT BE ADVANTAGEOUS TO COMPUTE SOME VARIABLES ONLY FOR THE WALL CENTRES OF THE CELLS USED FOR OTHER VARIABLES?
- THE OBVIOUS CANDIDATES FOR CELL-WALL STORAGE ARE THE VELOCITIES, WHICH APPEAR TO BE NEEDED ONLY FOR CELL WALLS.

HTE 1 15	3/15	THE STAGGERED GRID; DESCRIPTION.

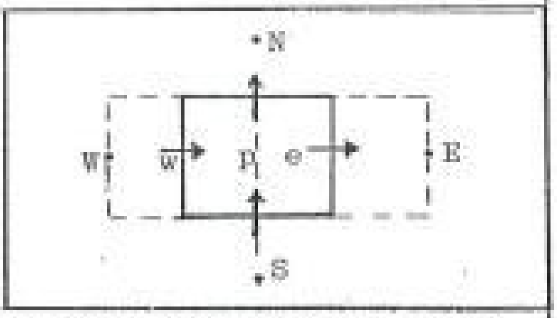
- ALL VARIABLES EXCEPT VELOCITY (SAY, u, v, w) ARE "STORED AT" THE POINTS P, N, S,... AS IN LECTURE 14.
- VALUES OF u ARE STORED AT WALL CENTRES w AND e, THOSE OF v AT s AND n, THOSE OF w AT l AND h.



- IN THE CASE OF NON-RECTANGULAR CELLS, IT MAY BE NECESSARY TO DISTINGUISH BETWEEN THE VELOCITY STORED AT A CELL-WALL CENTRE AND THE NORMAL COMPONENT WHICH GIVES THE MASS FLOW RATE.

HTE 1 15	4/15	THE STAGGERED GRID; ADVANTAGES.

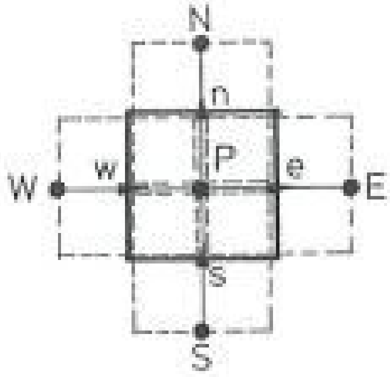
- ADVANTAGES FOR CALCULATION OF CONVECTION TERMS: THE u'S, v'S AND w'S NEEDED FOR MASS-FLOW CALCULATIONS ARE AVAILABLE AT THE RIGHT POINTS; THERE IS NO NEED FOR INTERPOLATION.
- ADVANTAGE FOR THE CALCULATION OF SOURCE TERMS IN THE MOMENTUM-EQUATION:
- THE VELOCITY u_w IS INFLUENCED BY THE PRESSURE DIFFERENCE $p_W - p_P$.
- u_e IS INFLUENCED BY $p_P - p_E$.



- THERE IS THEREFORE NO NEED FOR INTERPOLATION FOR THE EVALUATION OF MOMENTUM-SOURCE TERMS.

HTE 1 / 15	5/15	THE STAGGERED GRID; THE CONTROL VOLUMES FOR VELOCITIES.

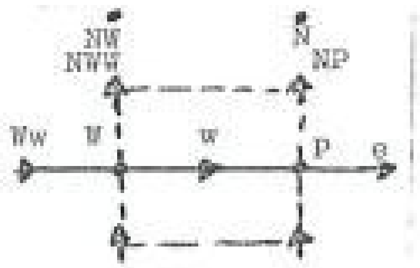
- THE SKETCH SHOWS THE FOUR CELLS SURROUNDING THE POINTS WHERE u_w, u_e, v_s, v_n ARE COMPUTED, FOR A 2D STRICT CARTESIAN GRID.
- THE VELOCITY CELLS (CONTROL VOLUMES) THUS FORM TWO ADDITIONAL OVERLAPPING SUBDIVISIONS OF THE INTEGRATION SPACE.



- IN A 3D PROBLEM THERE ARE FOUR DISTINCT CELL SYSTEMS: FOR u, v, w AND ALL OTHER VARIABLES.
- PRESSURES ARE STORED AT POINTS LYING ON THE WALLS OF VELOCITY CELLS.

HTE 1 / 15	6/15	A DISADVANTAGE OF THE STAGGERED-CELL SYSTEM

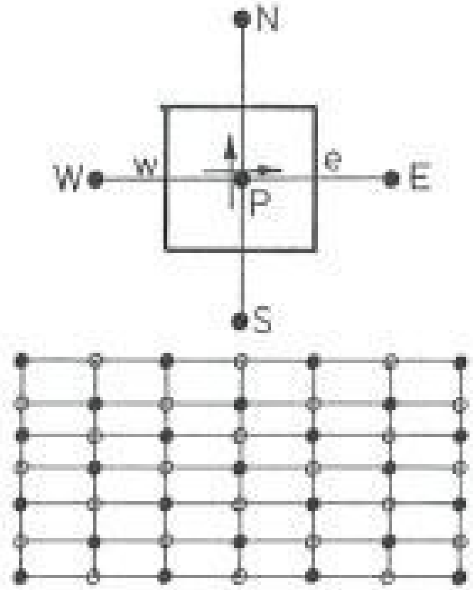
- THE NEED FOR CONVECTION TERMS IN THE CONSERVATION EQUATION FOR u:
  - u_w IS CALCULATED FROM THE BALANCE EQUATION FOR
  - SO u_W MUST BE TAKEN AS $\frac{1}{2}(u_w + u_{Ww})$.
  - SIMILARLY, v THROUGH THE NORTH WALL MUST BE TAKEN AS $\frac{1}{2}(v_{NWW} + v_{NP})$.



- THEREFORE INTERPOLATION IS NEEDED AFTER ALL.
- THE QUESTION THEN ARISES: WHY NOT ENJOY THE SIMPLICITY OF A SINGLE NON-STAGGERED GRID?

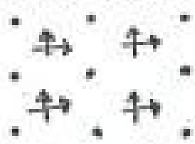
HTE 1 15	7/15	OBJECTION TO THE NON-STAGGERED GRID

- SUPPOSE u'S AND v'S ARE STORED AT SAME POINTS AS p'S.
- THEN PRESSURE DIFFERENCE DRIVING UP WILL BE $\frac{1}{2}(p_W + p_P) - \frac{1}{2}(p_P + p_E)$, I.E. $\frac{1}{2}(p_W - p_E)$.
- THEN IT IS POSSIBLE TO HAVE A PRESSURE FIELD IN WHICH •'S AND ○'S DIFFER MARKEDLY, BUT IN WHICH THERE IS NO MOMENTUM SOURCE WHATEVER.
- THIS IS UNREALISTIC.



HTE 1 15	8/15	DISCUSSION OF WHETHER TO ADOPT THE STAGGERED GRID.

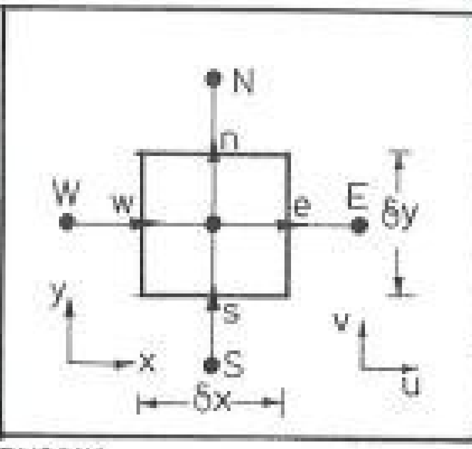
- THE DIFFICULTY WITH THE NON-STAGGERED GRID IS THAT IT CONTAINS EITHER TOO MUCH OR TOO LITTLE INFORMATION.
- EXAMPLE: AN ADDITIONAL (ARBITRARY) STATEMENT ABOUT THE LINKAGE BETWEEN •'S AND ○'S IS NEEDED IN ORDER TO AVOID THE UNREALISM OF THE PANEL 7 PRESSURE FIELD.
- THE NECESSITY TO USE p'S 2-GRID-INTERVALS APART IN PANEL 7 RATHER THAN 1-INTERVAL APART IN PANEL 6 MAKES THE FORMER LESS ACCURATE FOR GIVEN GRID FINENESS.
- THOUGH BOTH ARRANGEMENTS ARE POSSIBLE, MOST LARGE-SCALE USERS FAVOUR THE STAGGERED GRID.
- OTHER STAGGERED-GRID ARRANGEMENTS MAY HAVE (STILL-UNEXPLORED) MERITS, E.G:



HTE 1 15	9/15	THE CONTINUITY EQUATION FOR THE STAGGERED GRID (2D FOR SIMPLICITY).

- DEFINITIONS: LET SIDE LENGTHS BE δx, δy. LET TIME INTERVAL BE δt.
- CONTINUITY:

$$\frac{\delta x \delta y}{\delta t}(\rho_P - \rho_{P-}) = \left[(\rho u)_w - (\rho u)_e\right]\delta y + \left[(\rho v)_s - (\rho v)_n\right]\delta x + S_{mass}\,\delta x \delta y.$$



- COMMENT: • THE EXTENSION TO 3D IS OBVIOUS.
  - NON-CARTESIAN GRIDS REQUIRE MORE ELABORATE GEOMETRICAL DESCRIPTIONS.

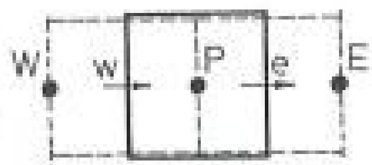
HTE 1 15	10/15	THE 2D CONTINUITY EQUATION; CORRECTION FORM.

- DEFINITIONS: • LET u_*, v_*, ρ_*, ETC. REPRESENT APPROXIMATIONS OR GUESSES FOR u, v, ρ.
  - LET $u' \equiv u - u_*$, WHERE $u \equiv$ THE TRUE VALUE; ETC.
  - LET $E_{mass}\,\delta x \delta y \equiv -\{\frac{\delta x \delta y}{\delta t}(\rho_{*P} - \rho_{P-}) + \delta y\left[(\rho_* u_*)_e - (\rho_* u_*)_w\right] + \delta x\left[(\rho_* v_*)_n - (\rho_* v_*)_s\right] - S_{*mass}\,\delta x \delta y\}$.
- CORRECTION FORM OF THE CONTINUITY EQUATION:

$$\frac{\delta x \delta y}{\delta t}\rho'_P + \left[(\rho u)'_e - (\rho u)'_w\right]\delta y + \left[(\rho v)'_n - (\rho v)'_s\right]\delta x - S'_{mass}\,\delta x \delta y = E_{mass}\,\delta x \delta y$$

- NOTE: THE 3D FORM IS EASILY DEVELOPED, BY EXTENSION.

HTE 1 15	11/15	THE 2D CORRECTION FORM OF CONTINUITY; USE OF PRESSURE AS LINKING VARIABLE.



- FOCUS OF ATTENTION (FOR SAKE OF EXAMPLE): $(\rho u)'_e$.
- WHAT INFLUENCES $(\rho u)'_e$?
  - $p'_P - p'_E$, THE PRESSURE GRADIENT.
  - u'_w, VIA CONVECTION OF MOMENTUM.
  - p'_e (AVERAGE (HOW WEIGHTED?) OF p'_P AND p'_E), VIA INFLUENCE ON DENSITY.
- DIRECTION OF PRESSURE INFLUENCE: INCREASING p'_P MUST INCREASE $(\rho u)_e$ AND ρ'_P, AND DECREASE $(\rho u)'_w$.
- CONSEQUENCE: ALL TERMS ON LHS OF PANEL 10 EQUATION WILL INCREASE WITH p'_P .

HTE 1 15	12/15	INFLUENCE OF PRESSURE CHANGES ON VELOCITY CHANGES.

- NOTE: THIS IS AN ADVANCE TREATMENT, CONCENTRATING ON THE MAIN IDEAS, OF WHAT WILL BE DISCUSSED IN DETAIL LATER.
- THE MOMENTUM EQUATIONS, DIFFERENTIATED, LEAD TO:

  $u'_e = c_{ew}\, u'_w + a_e(p'_P - p'_E)$

  $u'_w = c_{we}\, u'_e + a_w(p'_W - p'_P)$
- SEPARATION OF u'_e AND u'_w :

  $u'_e = \{a_e(p'_P - p'_E) + c_{ew}a_w\,(p'_W - p'_P)\}/(1 - c_{we}c_{ew})$

  $u'_w = \{a_w(p'_W - p'_P) + c_{we}\, a_e(p'_P - p'_E)\}/(1 - c_{we}c_{ew})$
- NOTE: $c_{ew} \to 1$ FOR $u > 0$, $\to 0$ FOR $u < 0$;

  $c_{we} \to 0$ FOR $u > 0$, $\to 1$ FOR $u < 0$.

HTE 1 15	$\frac{13}{15}$	INFLUENCES OF PRESSURE CHANGES ON ρu AND ρ_P CHANGES.

- NOTE: BECAUSE   WILL BE REPEATEDLY RE-CALCULATED, ONLY APPROXIMATE VALUES ARE NEEDED; SO COEFFICIENTS CAN BE APPROXIMATE.
- FOR THIS REASON, $(\rho u)'$ IS USUALLY REPLACED BY $\rho u'$, THE INFLUENCE OF p' ON $(\rho u)'$ BEING EXERTED ONLY VIA u'.
- THEN $(\rho u)'_e = a_e \rho_e (p'_P - p'_E)$,

  AND $(\rho u)'_w = a_w \rho_w (p'_W - p'_P)$,

  WITH THE c_{ew} AND c_{we} INFLUENCES ALSO IGNORED.
- THE INFLUENCE OF PRESSURE ON DENSITY IS TAKEN ACCOUNT OF IN THE FIRST TERM OF PANEL 10:

$$\frac{\delta x . \delta y .}{\delta t} \rho'_P = \frac{\delta x . \delta y .}{\delta t} \left(\frac{\partial \rho}{\partial p}\right)_P . p'_P .$$

HTE 1 15	$\frac{14}{15}$	PRESSURE CORRECTION, p', AS A LINKING VARIABLE IN THE CONTINUITY EQUATION.

- WITH THE APPROXIMATIONS OF PANEL 13, THE EQUATION OF PANEL 10 BECOMES:

$$p'_P \frac{\delta x . \delta y .}{\delta t} \left(\frac{\partial \rho}{\partial p}\right)_P + (a_e \rho_e + a_s \rho_w)\delta y + (a_n \rho_n + a_s \rho_s)\delta x =$$

$$= E_{mass}\, \delta x . \delta y + (p'_E\, a_e \rho_e + p'_W\, a_w \rho_w)\delta y$$

$$+ (p'_N\, a_n \rho_n + p'_S\, a_s \rho_s)\delta x ,$$

I.E. $\boxed{a_P p'_P = b_P + a_E p'_E + a_W p'_W + a_N p'_N + a_S p'_S}$.

- THIS IS A "POISSON EQUATION" FOR p' IN FINITE-DIFFERENCE FORM.
- IF IT IS SOLVED FOR ALL POINTS, AND THE CORRESPONDING VELOCITY AND DENSITY CORRECTIONS ARE MADE, CONTINUITY WILL BE SATISFIED FOR ALL CELLS.

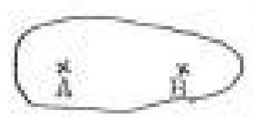
HTE 1 15	15/15	CONCLUDING REMARKS

- IN NUMERICAL SOLUTION PROCEDURES FOR THE HYDRODYNAMIC EQUATIONS, THE PRESSURE-CORRECTION EQUATION, BASED ON CONTINUITY, OCCUPIES A CENTRAL POSITION.
- THE COEFFICIENTS MUST ALWAYS BE SUCH THAT AN INCREASE IN PRESSURE CORRECTION TENDS TO FORCE FLUID FROM THE CELL.
- FOR COMPRESSIBLE FLOWS, THIS INFLUENCES HOW THE DENSITY INTERPOLATION IS CARRIED OUT.
- THE CONVECTIVE INTERACTIONS (c_{ew}, c_{we}) ARE IMPORTANT; BUT THEY ARE OFTEN IGNORED, SO THAT THE p' COEFFICIENTS FOR AN INTER-NODE LINK ARE THE SAME FOR BOTH THE LINKED NODES.

HTE 1 16	1/15	PART IV. CLASSIFICATION OF PROBLEMS AND PROCEDURES LECTURE 16. PATTERNS OF INFLUENCE

- CONTENTS:
  - KINDS OF INFLUENCE.
  - REPRESENTATION BY THE FDE AND ITS COEFFICIENTS.
  - ONE-WAY INFLUENCE PATTERNS.
  - THE PRESSURE-CORRECTION EQUATION.
  - PRESSURE-TRANSMITTED INFLUENCES.
  - RADIATION.
  - NOMENCLATURE: ELLIPTIC, PARABOLIC, HYPERBOLIC.

HTE 1 16	2/15	KINDS OF INFLUENCE OF ONE PART OF A FLOW ON ANOTHER; GENERAL.



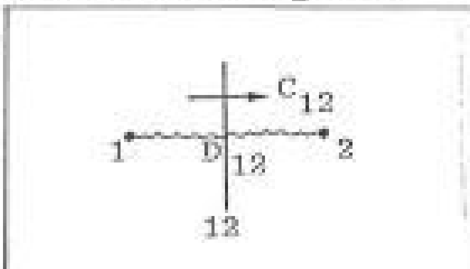
- DEFINITION: INFLUENCE TRAVELS FROM A TO B IN A FLOW IF, WITH THE PRESCRIBED BOUNDARY CONDITIONS, A CHANGE OF PROPERTY AT A (E.G. THE RESULT OF A CHANGED SOURCE STRENGTH) LEADS TO A CHANGE OF PROPERTIES AT B.
- EXAMPLES:
  - IF GENERAL FLOW DIRECTION IS FROM A TO B, A HEAT SOURCE AT A WILL CAUSE A TEMPERATURE RISE AT B.
  - HOWEVER, A HEAT SOURCE AT B MAY HAVE LITTLE INFLUENCE ON THE TEMPERATURE AT A.
- NOTES:
  - INFLUENCES ARE NOT ALWAYS MUTUAL.
  - THE <u>STRENGTH</u> OF INFLUENCE IS IMPORTANT.

HTE 1 16	3/15	SIX KINDS OF INFLUENCE; PHYSICAL NATURE.

- HISTORICAL: EVENTS AT EARLIER TIME INFLUENCE THOSE AT LATER TIME. THE REVERSE IS NOT TRUE.
- CONVECTIVE: UPSTREAM EVENTS INFLUENCE DOWNSTREAM PROPERTIES. THE REVERSE IS NOT TRUE.
- DIFFUSIVE: DIFFUSION (HEAT CONDUCTION, VISCOUS ACTION) SPREADS INFLUENCE IN ALL DIRECTIONS IN SPACE.
- RADIATIVE: AS FOR DIFFUSIVE.
- PRESSURE: PRESSURE CHANGES SPREAD INFLUENCES IN ALL DIRECTIONS (EXCEPT IN SUPERSONIC FLOW).
- FLOW-RATE SPECIFICATION: THIS IS A DISTINCT KIND OF INFLUENCE, TO BE DESCRIBED LATER.

HTE 1 16	4/15	REPRESENTATION OF INFLUENCES BY THE FINITE-DIFFERENCE EQUATIONS.

- EQUATION: $\phi_P = \dfrac{(a + c_{P-}\phi_{P-} + c_W\phi_W + c_E\phi_E + \dots)}{(-b + c_{P-} + c_W + c_E + \dots)}$
- CONDITION FOR FINITE INFLUENCE: INFLUENCE FLOWS FROM POINT i TO POINT P IF, IN THIS EQUATION, COEFFICIENT $c_i > 0$. OTHERWISE THERE IS NO INFLUENCE.
- MUTUALITY:
  - LET c_{W2} MEAN c_W WHEN P IS AT 2; LET c_{E1} MEAN c_E WHEN P IS AT 1.
  - THEN THE ANALYSIS OF LECTURE 15 IMPLIES: $c_{W2} = D_{12} + f_{12}C_{12}$; $c_{E1} = D_{12} + (f_{12} - 1)C_{12}$, WHERE $f_{12} = \frac{1}{2}$ FOR CENTRAL DIFFERENCES; IN GENERAL $f_{12} = f_{12}\left(\frac{C_{12}}{D_{12}}\right)$.

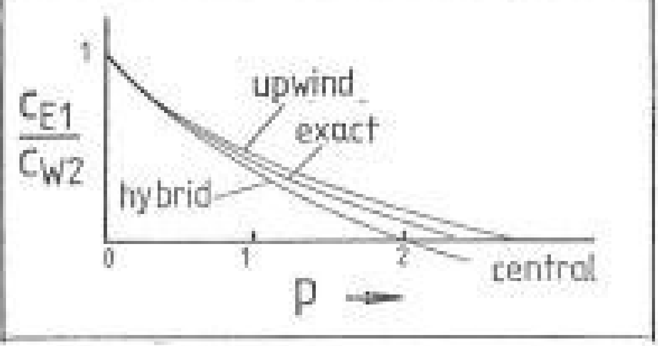


HTE 1 16	5 15	REPRESENTATION OF INFLUENCES BY FD COEFFICIENTS; MUTUALITY (CONTINUED).

- DISCUSSION:
  - $c_{W2} = c_{E1}$ MEANS 1~2 INFLUENCE IS MUTUAL
  - IN GENERAL, $c_{W2} - c_{E1} - C_{12} \neq 0$.
  - SO $c_{W2} = c_{E1}$ ONLY WHEN CONVECTION IS ABSENT.
- EFFECT OF PECLET NO. ON COEFFICIENT RATIO:
  - LET $P \equiv C_{12}/D_{12}$.
  - CONSIDER c_{E1}/c_{W2} FOR $P > 0$ (SAME AS c_{W2}/c_{E1} FOR $P < 0$).
  - UPWIND FORMULA: $c_{E1}/c_{W2} = 1/(1+P)$.
  - EXACT FORMULA:

    $$c_{E1}/c_{W2} = \exp(-P).$$
  - HYBRID:

    $$\frac{c_{E1}}{c_{W2}} = \max\left[0,\ \frac{1-\frac{1}{2}P}{1+\frac{1}{2}P}\right]$$
  - CONCLUSION.



HTE 1 16	6 15	REPRESENTATION OF INFLUENCES BY FD COEFFICIENTS; MUTUALITY (CONTINUED).

- CONCLUSIONS ABOUT DIFFUSION AND CONVECTION:
  - WHEN P IS SMALL, INFLUENCES BETWEEN NEIGHBOUR POINTS ARE MUTUAL.
  - WHEN P IS LARGE, INFLUENCE IN THE UPSTREAM DIRECTION IS VERY SMALL.
- TIME DEPENDENCE:
  - UNLESS δt IS VERY LARGE, c_{P-} CARRIES FINITE INFLUENCE FROM EARLIER TIMES TO LATER TIMES.
  - IN THE EQUATION FOR ϕ_{P-} (I.E. ϕ_P AT THE EARLIER TIME) THERE IS NO LINK WITH ϕ_P.
  - IN THE EQUATION FOR ϕ_P THERE IS NO LINK WITH ϕ_{P+} (I.E. ϕ_P AT THE LATER TIME).
  - TIME INFLUENCES ARE THEREFORE <u>NOT</u> MUTUAL; THEY ACT LIKE CONVECTIVE INFLUENCES, IN ONLY ONE WAY.

HTE 1 16	7/15	THE IMPORTANCE OF ONE-WAY PATTERNS OF INFLUENCE.

- OCCURRENCE: • A TIME-DEPENDENT FLOW HAS HISTORICAL INFLUENCES FLOWING ONLY IN ONE DIRECTION.
  - A "STRAIGHT-THROUGH" STEADY FLOW AT HIGH REYNOLDS NO. ($P >> 1$) (E.G. FLOW THROUGH A DUCT WITHOUT RECIRCULATION) HAS NEGLIGIBLE DIFFUSIVE INFLUENCE FROM DOWNSTREAM; AND EFFECTS OF RADIATION, PRESSURE AND FLOW-RATE SPECIFICATION MAY BE NEGLIGIBLE.
- CONSEQUENCES: • SOLUTION OF THE FDE'S MAY TAKE PLACE SEQUENTIALLY IN ONE "MARCHING INTEGRATION".
  - THE DIMENSIONALITY OF COMPUTER STORAGE IS LESS BY ONE DIMENSION THAN THAT OF THE FLOW PHENOMENON.

HTE 1 16	8/15	INFLUENCES BY WAY OF PRESSURE; THE PRESSURE-CORRECTION EQUATION.

- EQUATION FOR p'_P: THE CONTINUITY EQUATION CAN BE PUT IN THE SAME FORM AS THAT OF PANEL 4, WITH APPROPRIATE DEFINITIONS. THUS:

$$p'_P = \frac{(a + c_W p'_W + c_E p'_E + \ldots (N,S,H,L))}{(-b + c_P)}$$

- NOTES: • p'_{P-} IS MISSING FROM THE NUMERATOR BECAUSE THERE IS NO QUESTION OF CORRECTING A PAST PRESSURE; SO $p'_{P-} = 0$.
  - $a \equiv$ THE ERROR IN THE CONTINUITY EQUATION FOR THE CELL, I.E. THE EXCESS OF INFLOW OVER OUTFLOW PLUS INCREASE IN MASS CONTENT.
  - THE MEANINGS OF THE c'S MUST BE EXAMINED.

HTE 1 16	9/15	COEFFICIENTS IN THE PRESSURE-CORRECTION EQUATION.

- SIMPLEST CASE: • NEGLECT EFFECTS OF MOMENTUM CONVECTION AND DENSITY CHANGE IN ρu'S.
  - THEN, INFLOW RATE THROUGH w FACE INCREASES BY:

$$c_W \, (p'_W - p'_P)$$

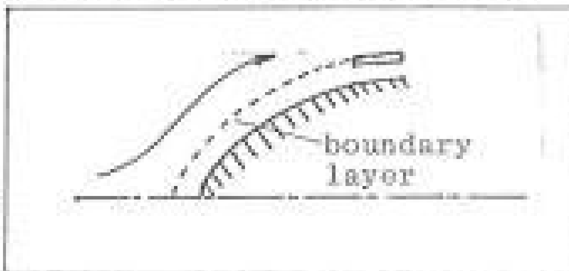
  - INFLOW RATE THROUGH e INCREASES BY $c_E(p'_E - p'_P)$.
  - THEN $c_P = V\left(\frac{\partial\rho}{\partial p}\right)_P + c_W + c_E + c_N + c_S + c_H + c_L$.
  - ALSO $b \equiv \frac{\partial}{\partial p'_P} \, S_{mass} . V$.
- CONSEQUENCES: • c'S ARE ALL POSITIVE.
  - TYPICALLY, $c_W \approx A_w \Big/ \left| \sqrt{u^2+v^2+w^2} \right|$
  - PRESSURE INFLUENCES SPREAD IN ALL DIRECTIONS.

HTE 1 16	10/15	COEFFICIENTS IN THE PRESSURE-CORRECTION EQUATION; REFINEMENTS.

- ACCOUNTING FOR p' IN $(\rho u)'$: • FOR COMPLETE ANALYSIS, IT IS NECESSARY TO COMBINE RESULTS OF PANELS 15-13 AND 15-15.
  - α_e REMAINS TO BE DETERMINED. THIS MUST BE CHOSEN SO THAT $(\rho u)'_e$ IS INDEPENDENT OF p'_E WHEN u_e > SOUND VELOCITY; AND INDEPENDENT OF p'_P, WHEN $-u_E$ > SOUND VELOCITY.
  - DETAILS WILL BE GIVEN LATER.
  - RESULT: FDE COEFFICIENTS FALL TO ZERO WHEN MACH NO. EXCEEDS 1, PREVENTING INFLUENCES FROM TRAVELLING UPSTREAM.
- ACCOUNTING FOR MOMENTUM-CONVECTION TERMS:
  - THESE DO NOT INFLUENCE THE MATTER.

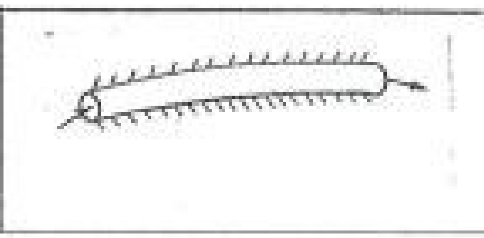
HTE 1 / 16	11 / 15	CONCLUSIONS ABOUT INFLUENCES SPREAD BY PRESSURE EFFECTS.

- GENERAL CASE: THROUGH PRESSURE, AND THE NECESSITY TO SATISFY CONTINUITY, CHANGES AT A TRANSMIT THEIR EFFECTS TO B REGARDLESS OF POSITION.
- SUPERSONIC CASE: THIS DOES NOT OCCUR, HOWEVER, WHEN B IS UPSTREAM OF A AND THE VELOCITY IS SUPERSONIC B.
- LONG THIN FLOWS: BECAUSE OF THE A_w IN THE C_w EXPRESSION OF PANEL 9, PRESSURE CHANGES HAVE MORE EFFECT THROUGH LARGE THAN SMALL AREAS.
  - THEREFORE, FOR LONG THIN FLOWS, LATERAL PRESSURE RELIEF IS EASY.
  - IN THESE, LONGITUDINAL INFLUENCES OF PRESSURE ARE SMALL.



HTE 1 / 16	12 / 15	PRESSURE ~ FLOW-RATE INTERACTIONS IN CONFINED FLOWS; TWO CASES.

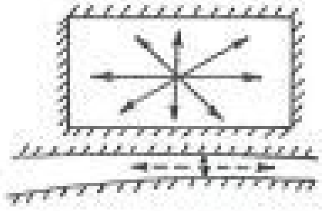
- COMMON FEATURES:
  - CONFINEMENT IN A LONG THIN DUCT.
  - NO RECIRCULATION (REVERSE FLOW).



- FIRST CASE: THE FLOW RATE IS FIXED; THE PRESSURE DIFFERENCE FROM ENTRY TO EXIT IS ONE OF THE QUANTITIES TO BE DETERMINED.
- SECOND CASE: THE PRESSURE DIFFERENCE IS FIXED; THE FLOW RATE IS TO BE DETERMINED.
- IMPORTANT DIFFERENCE: • IN THE FIRST CASE, INFLUENCES TRAVEL ONLY DOWNSTREAM.
  - IN THE SECOND THEY CAN TRAVEL UPSTREAM ALSO.

HTE 1 16	13/15	RADIATIVE INFLUENCES; TWO CASES.

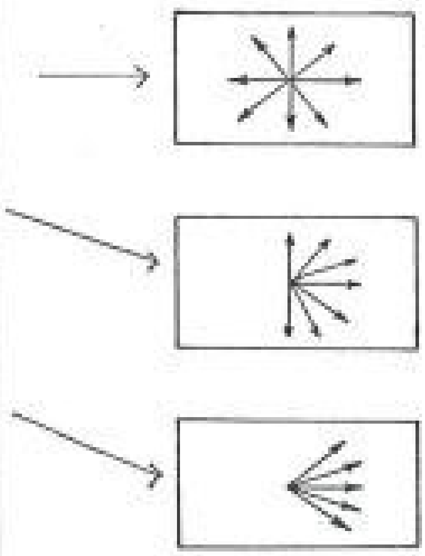
- GENERAL CASE: BECAUSE RADIATION TRAVELS TO EVERY POINT THAT CAN BE SEEN, ITS INFLUENCE PATTERN IS EXTENSIVE.



- LONG THIN REGIONS: THE LATERAL GRADIENTS MAY BE MUCH STEEPER THAN THE LONGITUDINAL ONES. THEN LONGITUDINAL INFLUENCES MAY BE NEGLIGIBLE.
- NOTES: • INFLUENCES ARE MOST EXTENSIVE FOR MODERATELY TRANSPARENT MEDIA.
  - FOR HIGHLY ABSORBING ONES, THE INFLUENCE PATTERN IS AS FOR CONDUCTION.

HTE 1 16	14/15	NAMES FOR THE VARIOUS INFLUENCE PATTERNS

- ELLIPTIC: INFLUENCES SPREAD IN ALL DIRECTIONS.
- PARABOLIC: INFLUENCES SPREAD ONLY "DOWNSTREAM" (IN SPACE OR TIME).
- HYPERBOLIC: INFLUENCES SPREAD ONLY INTO A WEDGE OR CONE, WITH ANGLE DETERMINED BY THE SPEED OF SOUND.
- NOTE: MANY FLOWS ARE OF MIXED TYPE.



HTE 1 16	15/15	FINAL REMARKS ABOUT INFLUENCE PATTERNS.

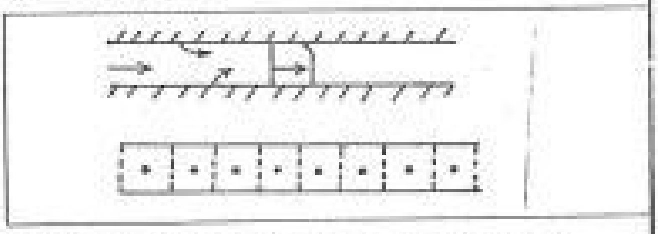
- RECOGNITION OF THE INFLUENCE PATTERN OF A PROCESS TO BE INVESTIGATED IS AN ESSENTIAL PRE-REQUISITE FOR THE CORRECT CHOICE OF SOLUTION PROCEDURE.
- THE EASIEST PROBLEMS ARE THOSE POSSESSING A ONE-WAY CHARACTER. THUS 2D PARABOLIC IS EASIER TO SOLVE THAN 2D ELLIPTIC; 3D HYPERBOLIC IS EASIER THAN 3D ELLIPTIC; ETC.
- OF THE SIX KINDS OF INFLUENCE MENTIONED AT THE START: THE MOST COMMONLY CONSIDERED ARE:
  - HISTORICAL,
  - CONVECTIVE,
  - DIFFUSIVE,
  - PRESSURE-TRANSMITTED.

HTE 1 17	1/15	LECTURE 17. CLASSIFICATION OF FLOW PROBLEMS.

- CONTENTS:
  - PARABOLIC IN SPACE.
  - PARABOLIC IN TIME.
  - HYPERBOLIC IN SPACE.
  - HYPERBOLIC IN TIME.
  - ELLIPTIC IN SPACE.
- NOTES:
  - THERE IS NO QUESTION OF A PROCESS WHICH IS "ELLIPTIC IN TIME".
  - PROCESSES CAN ALSO BE CLASSIFIED AS 1D, 2D, 3D, 4D.
  - THE WORDS "PARABOLIC", "ELLIPTIC", "HYPERBOLIC" ARE BORROWED FROM THE THEORY OF DIFFERENTIAL EQUATIONS; BUT THIS CONNEXION IS NOT ESSENTIAL.

HTE 1 17	2/15	PARABOLIC PROCESSES; ONE-DIMENSIONAL.

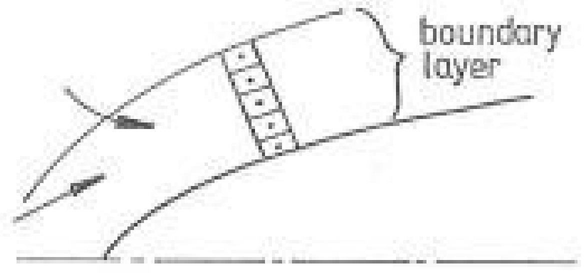
- EXAMPLE WITH DISTANCE AS THE VARIABLE:
  - STEADY PIPE FLOW, WITH
    - VARIATIONS IN SECTION SMALL COMPARED WITH THOSE BETWEEN FLUID AND WALL.



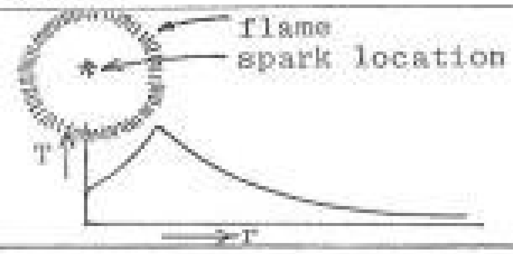
  - 1D CELL ARRANGEMENT.
  - MARCHING INTEGRATION FROM LEFT.
  - NOTE: FLOW RATE MUST BE SPECIFIED.
- EXAMPLE WITH TIME AS THE VARIABLE:
  - WELL-STIRRED LAKE, CHANGING IN TEMPERATURE AS A RESULT OF DIURNAL CHANGES IN ATMOSPHERIC TEMPERATURE.
- NOTE: ONLY <u>ONE</u> STORAGE POINT IS NEEDED, FOR EACH DEPENDENT VARIABLE; COMPUTER STORAGE IS ZERO-DIMENSIONAL.

HTE 1 17	3/15	PARABOLIC PROCESSES; TWO-DIMENSIONAL EXAMPLES.

- TWO DISTANCE VARIABLES:
  - STEADY FLOW AT HIGH RE ON SYMMETRICAL BODY.
  - ϕ'S VARY APPRECIABLY ACROSS THE LAYER.
  - AT ANY POINT IN THE CALCULATION, A STRING OF CELLS ACROSS THE LAYER IS UNDER CONSIDERATION.



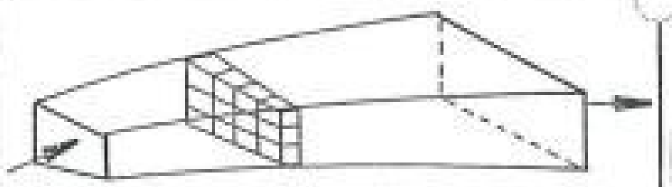
- ONE DISTANCE AND ONE TIME VARIABLE:
  - FLAME PROPAGATION FROM A SPARK IN A GAS AT REST.



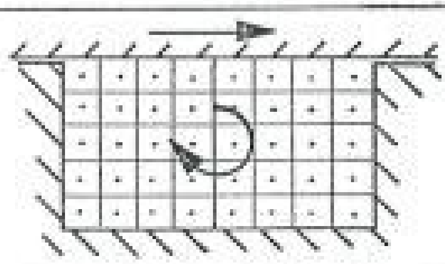
- NOTE: • PROCESS IS 2D.
  - COMPUTER STORAGE IS 1D, FOR EACH VARIABLE.

HTE 1 17	4/15	PARABOLIC PROCESSES; THREE-DIMENSIONAL EXAMPLES.

- THREE DISTANCE VARIABLES:
  - STEADY FLOW IN A MILDLY-CURVED DUCT.
  - NO RECIRCULATION.
  - AXIAL CONDUCTION ETC. UNIMPORTANT.
  - AT ANY POINT IN THE CALCULATION, THE CELLS UNDER CONSIDERATION OCCUPY A SLAB, SUB-DIVIDED TWO WAYS.



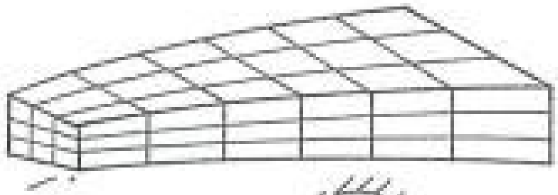
- TWO DISTANCE VARIABLES, AND TIME:
  - THE MOVABLE COVER STRIP* IS SUDDENLY SET IN MOTION.
  - THE FLUID STARTS TO CIRCULATE.

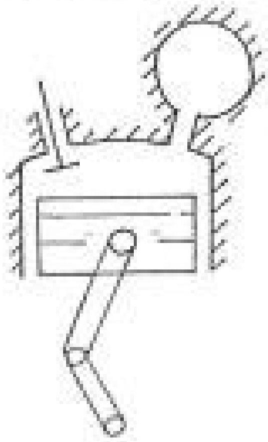


- NOTE: • PROCESS IS 3D.
  - COMPUTER STORAGE IS 2D, FOR EACH VARIABLE.

HTE 1	5	PARABOLIC PROCESSES;
17	15	FOUR-DIMENSIONAL EXAMPLES.

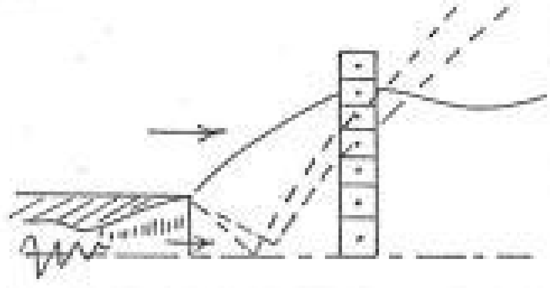
- PARABOLIC IN BOTH SPACE AND TIME: • UNSTEADY FLOW IN DUCT OF PANEL 4.
  - A 3D CELL ARRANGEMENT IS NEEDED.
- THE GENERAL CASE: • FLOW AND FLAME PROPAGATION IN A RECIPROCATING-ENGINE CYLINDER.
- NOTE: IN BOTH CASES, THE PROBLEM IS 4D, AND THE COMPUTER STORAGE 3D, FOR EACH VARIABLE.



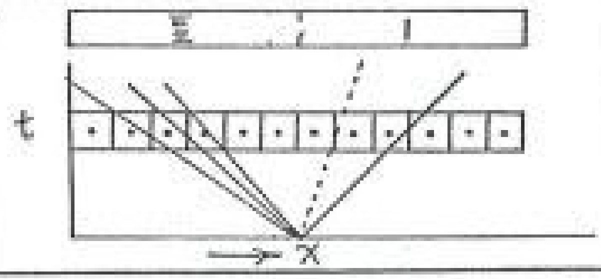


HTE 1	6	HYPERBOLIC PROCESSES;
17	15	TWO-DIMENSIONAL (N.B. 1D DO NOT EXIST).

- TWO DISTANCE VARIABLES:
  - ROCKET EXHAUST IN SUPERSONIC FLIGHT.
  - STEADY COMPRESSIBLE FLOW.
  - INTEGRATION PROCEEDS FROM UPSTREAM TO DOWN-STREAM, FOLLOWING A MOVING 1D STRING OF CELLS.

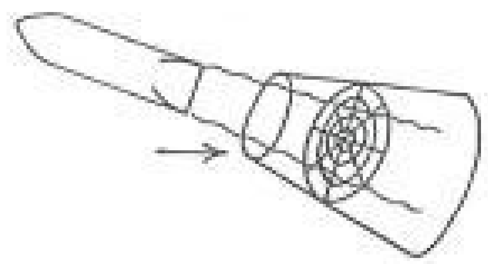


- ONE DISTANCE VARIABLE, AND TIME:
  - THE SHOCK TUBE AFTER BREAKAGE OF DIAPHRAGM.
  - THE STRING OF CELLS IS ALL AT FIXED TIME.
- NOTES: • COMPUTER STORAGE 1D.
  - SITUATION SIMILAR TO PARABOLIC.

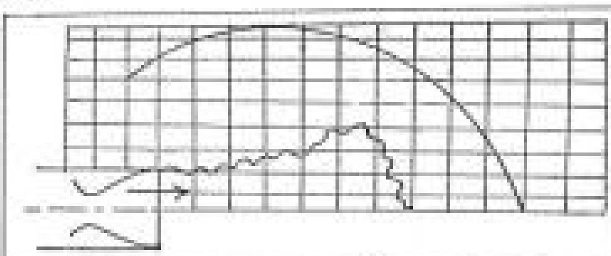


HTE 1 17	7/15	HYPERBOLIC PROCESSES: THREE-DIMENSIONAL.

- THREE SPACE VARIABLES:
  - ROCKET EXHAUST INCLINED TO LINE OF FLIGHT, STEADY.
  - AT ANY POINT IN THE SOLUTION, ATTENTION IS CONFINED TO A DISC-SHAPED SLICE. THIS IS MOVED DOWNSTREAM AS INTEGRATION PROCEEDS.

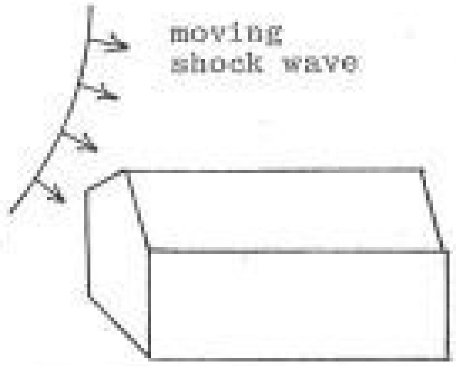


- TWO SPACE VARIABLES AND TIME:
  - THE START-UP PROCESS OF A ROCKET EXHAUST, AXI-SYMMETRICAL.
- NOTE: • PROCESS 3D.
  - STORAGE 2D.



HTE 1 17	8/15	HYPERBOLIC PROCESSES: FOUR-DIMENSIONAL.

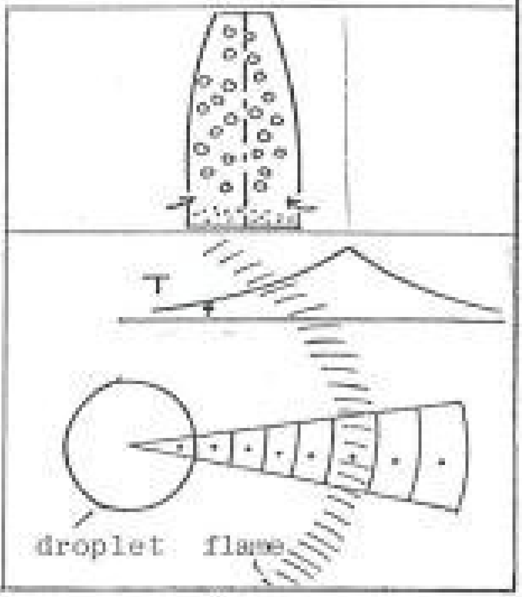
- EXAMPLE:
  - BLAST WAVE IMPINGES ON A GENERAL 3D BODY.
  - PROBLEM IS TO DETERMINE PRESSURES, FLOW RATES, ETC.
  - A 3D CELL ARRANGEMENT IS NEEDED.



- GENERAL REMARKS ABOUT HYPERBOLIC PROCESSES.
  - IN RESPECT OF STORAGE AND SOLUTION ARRANGEMENT THEY DO NOT DIFFER APPRECIABLY FROM PARABOLIC.
  - THE DIFFERENCES APPEAR IN THE DETAILS OF THE FINITE-DIFFERENCE FORMULAE, ESPECIALLY FOR ρ.

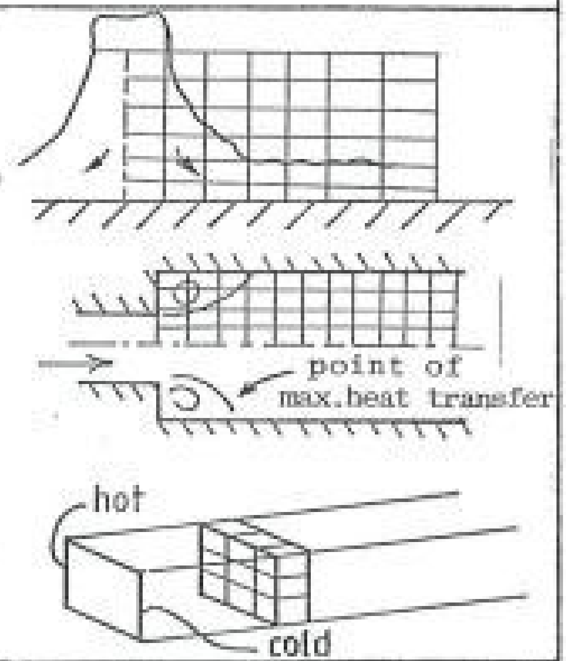
HTE 1 17	9/15	ELLIPTIC PROCESSES; ONE-DIMENSIONAL.

- GENERAL: ALL ELLIPTIC PROCESSES ARE STEADY.
- EX.1: FLOW IN BLAST FURNACE, WITH THE PRESUMPTION OF UNIFORMITY WITH RESPECT TO RADIAL AND ANGULAR POSITION.
- EX.2: STEADY BURNING OF A LIQUID-FUEL DROPLET IN A STAGNANT ATMOSPHERE UNDER CONDITIONS OF NO GRAVITY.
- EX.3: THE FIRST PROCESS OF PANEL 2, WITH PRESSURE DIFFERENCE FIXED RATHER THAN FLOW RATE.



HTE 1 17	10/15	ELLIPTIC PROCESSES; TWO-DIMENSIONAL.

- EX.1: IMPINGEMENT OF A JET BLOWING VERTICALLY DOWNWARD ON TO A PLANE SURFACE, E.G. TO EFFECT HEATING OR COOLING.
- EX.2: FLOW DOWNSTREAM OF A SUDDEN ENLARGEMENT; TURBULENT; PERHAPS WITH CHEMICAL REACTION.
- EX.3: FULLY DEVELOPED FLOW IN A DUCT OF RECTANGULAR SECTION, PERHAPS WITH SIDES OF DIFFERENT TEMPERATURE.

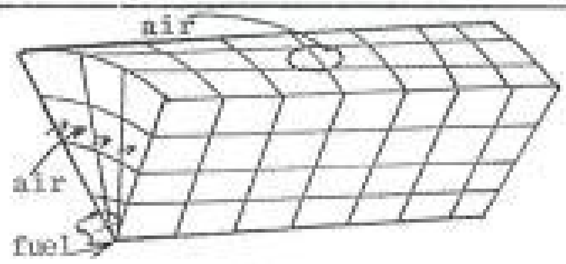


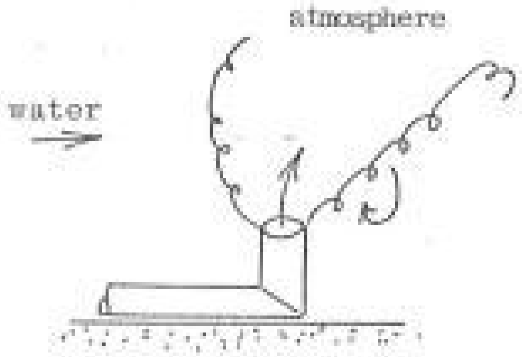
HTE 1 17	11 15	ELLIPTIC PROCESSES; THREE-DIMENSIONAL.

EX.1: • SECTOR THROUGH A COMBUSTOR OF A GAS-TURBINE ENGINE.
• 3D CELL SYSTEM NEEDED.

EX.2: • INJECTION OF WARM WATER FROM LAKE BOTTOM WITH A LATERAL CURRENT.

EX.3: • FLOW IMMEDIATELY DOWNSTREAM OF A COOLING TOWER.



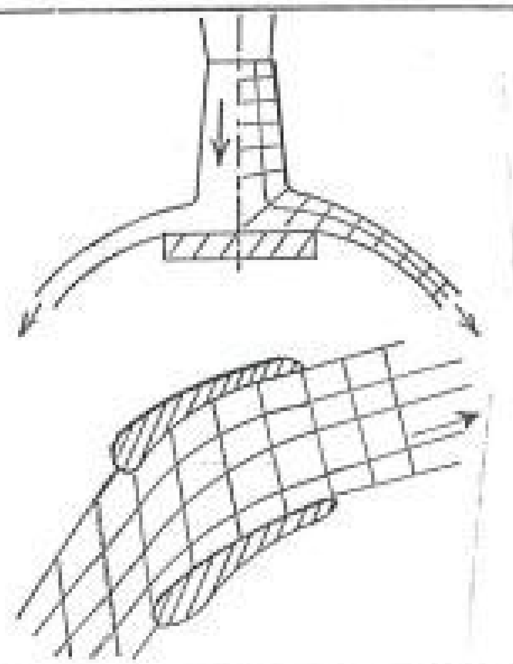


HTE 1 17	12 15	PARTIALLY-PARABOLIC PROCESSES; DEFINITION AND PROPERTIES.

- DEFINITION: A PARTIALLY-PARABOLIC PROCESS IS ONE IN WHICH CONVECTIVE AND DIFFUSIVE EFFECTS HAVE ONLY ONE-WAY INFLUENCES, THE FLOW IS STEADY, BUT THERE IS SUFFICIENT CURVATURE FOR PRESSURE INFLUENCES TO SPREAD UPSTREAM.
- PROPERTIES: • ITERATIVE SOLUTION IS NEEDED.
  - HOWEVER, THIS MAY PROCEED BY MARCHING SWEEPS IN THE DOWNSTREAM DIRECTION.
  - THEREFORE <u>ONLY</u> PRESSURE NEEDS STORAGE DIMENSIONALITY EQUAL TO THAT OF THE SPACE DOMAIN.

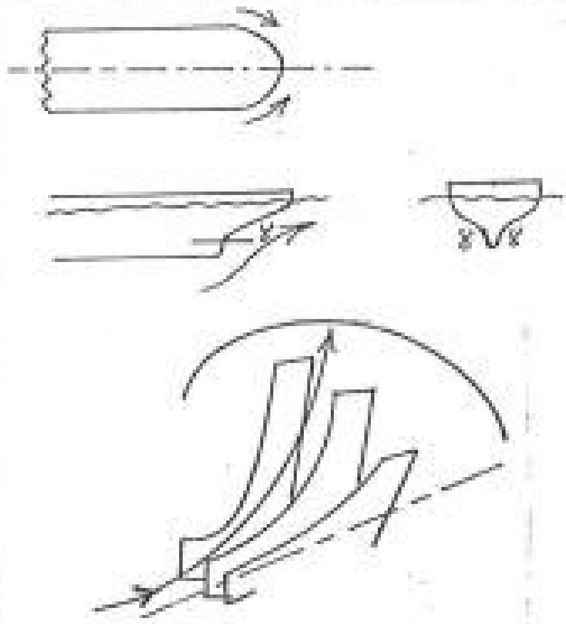
HTE 1 / 17	13 / 15	PARTIALLY-PARABOLIC PROCESSES; 2D EXAMPLES.

- <u>EX.1</u>: • FLOW OF WATER ON TO A PLATE, WITH INFLUENCES OF SURFACE TENSION AND GRAVITY.
  - A STREAM-LINE COORDINATE SYSTEM WILL BE CONVENIENT.
- <u>EX.2</u>: • 2D FLOW IN A TURBINE PASSAGE.
  - THE STREAM-LINE SYSTEM IS AGAIN CONVENIENT.
  - PRESCRIBED FIXED GRIDS CAN ALSO BE USED.



HTE 1 / 17	14 / 15	PARTIALLY-PARABOLIC PROCESSES; 3D EXAMPLES.

- <u>EX.1</u>: • FLOW IN THE VICINITY OF A SHIP'S STERN.
  - STRONG CURVATURE CAUSES LONGITUDINAL PRESSURE-INFLUENCE TRANSMISSION.
  - RECIRCULATION IS ABSENT (OR SHOULD BE).
- <u>EX.2</u>: • FLOW IN A CENTRIFUGAL-COMPRESSOR IMPELLOR.
- <u>NOTE</u>: IN BOTH CASES PRESSURE STORAGE IS 3D, THAT OF u, v, w, ETC., IS 2D.



HTE 1 17	15/15	FINAL REMARKS

- THE DISTINCTIONS BETWEEN PARABOLIC/HYPERBOLIC, ELLIPTIC AND PARTIALLY-PARABOLIC ARE EXTREMELY IMPORTANT; UNDERSTANDING THEM PERMITS CORRECT CHOICES OF METHODS TO BE MADE.
- THE PARTIALLY-PARABOLIC SYSTEMS ARE THE LEAST EXPLORED: BUT EARLY EXPERIENCE HAS BEEN ENCOURAGING (PRATAP AND SPALDING, 1975).
- MIXED PROCESSES ARE ALSO COMMON, E.G. THE COMPRESSOR PASSAGE WITH A SMALL REGION OF RECIRCULATION. THEY REQUIRE MIXED METHODS.

HTE 1 18	1 — 15	LECTURE 18. EXPLICIT SOLUTION PROCEDURES.

- CONTENTS:
  - GENERAL FEATURES OF EXPLICIT PROCEDURES.
  - HEAT CONDUCTION IN ONE DIMENSION.
  - AMPLITUDE RATIO; PHASE SHIFT.
  - HYDRODYNAMICS IN ONE DIMENSION.
  - INFLUENCES OF REYNOLDS NUMBER AND MACH NUMBER ON TIME-STEP LIMIT.
- NOTES: • EXPLICIT PROCEDURES ARE APPLICABLE ONLY TO TRANSIENT PROCESSES.
  - STEADY-STATE PROCESSES MAY BE ANALYSED BY ANALYSING THE RESULT OF MANY TIME INTERVALS.

HTE 1 18	2 — 15	GENERAL FEATURES OF EXPLICIT SOLUTION PROCEDURES.

- THE FDE (IN PLACE OF THAT OF PANEL 14.14):

$$\phi_P = \phi_{P-} + \left[a+b\,\phi_{P-} + \sum_{i=N,S,E,W,H,L} c_i(\phi_{i-}-\phi_{P-})\right]/c_{P-}$$

- SIGNIFICANCE: THE SOURCE, DIFFUSION AND CONVECTION PROCESSES FOR THE CELL HAVE THE SAME VALUE THROUGHOUT THE TIME INTERVAL AS AT THE BEGINNING.
- PROCEDURE: • AT TIME 0 , ALL ϕ'S ARE KNOWN (I.E. ϕ_{P-}, ϕ_{N-}, ϕ_{W-}, ...).
  - ALL ϕ_P'S FOR TIME t CAN THEREFORE BE EVALUATED BY EXPLICIT EVALUATION OF THE FDE FOR EACH POINT.
  - ANY ORDERLY EVALUATION PATTERN CAN BE USED.

HTE 1 18	$\frac{3}{15}$	HEAT CONDUCTION; A 2D PARABOLIC PROBLEM.

- THE PROBLEM: SUPPOSE THAT, AT $t = o$, THE TEMPERATURE DISTRIBUTION ($\phi \equiv T$) IS GIVEN BY: $\phi = \phi_o \sin(x/\lambda)$, AND THAT THE FLUID, OF UNIFORM Γ, c AND ρ MOVES AT UNIFORM VELOCITY u. THERE IS NO SOURCE.
- THE ANALYTICAL SOLUTION: SUBSTITUTION IN THE DIFFERENTIAL EQUATION VERIFIES THE SOLUTION:

$$\phi = \Phi \sin[(x - ut)/\lambda]; \; \Phi = \phi_o \exp\left(-\frac{\Gamma t}{\rho\lambda^2}\right).$$

- DISCUSSION: • THIS REPRESENTS A WAVE WHICH MOVES $u\,\delta t$ TO THE RIGHT IN THE TIME INTERVAL, t.
- ITS AMPLITUDE DECAYS BY FACTOR $\exp\left(\frac{-\Gamma\;t}{\rho\lambda^2}\right)$.

HTE 1 18	$\frac{4}{15}$	HEAT CONDUCTION; THE FINITE-DIFFERENCE SOLUTION FOR ONE TIME INTERVAL.

- FDE: $\phi_P = (1 - w - e)\,\phi_{P-} + w\phi_{W-} + e\phi_{E-}$, WHERE $w \equiv c_W/c_{P-}$, AND $e \equiv c_E/c_{P-}$.
- CONDITIONS AT BEGINNING OF TIME INTERVAL (TIME = O):
  LET: $\phi_{P-} = \phi_o \sin\xi$, $\phi_{W-} = \phi_o \sin(\xi-\delta)$, $\phi_{E-} = \phi_o \sin(\xi+\delta)$,
  WHERE: $\xi \equiv x_P/\lambda$, $\delta \equiv (x_P - x_W)/\lambda$.
- CONDITIONS AT THE END OF THE TIME INTERVAL: IT MAY BE SHOWN, BY SUBSTITUTION FOR ϕ_{P-}, ϕ_{W-}, ϕ_{E-} IN THE FDE, THAT:
  $\boxed{\phi_P = \Phi \sin(\xi+\varepsilon)}$, WHERE

$$\tan\varepsilon = \frac{(e-w)\sin\delta}{1-(e+w)(1-\cos\delta)}, \text{ AND } \frac{\Phi}{\phi_o} = \frac{1-(e+w)(1-\cos\delta)}{\cos\varepsilon}.$$

- SO FDE INDICATES THAT ϕ RETAINS SINE-CURVE FORM, SHIFTS IN X-DIRECTION IN PROPORTION TO ε, AND DECAYS IN AMPLITUDE BY AMOUNT DEPENDING ON e, w, δ, c.

HTE 1	5	HEAT CONDUCTION. DISCUSSION:
18	15	THE CHANGE IN PROPAGATION SPEED.

- $\varepsilon \equiv$ (x shift)/λ, ε EQUALS $u\ \ t/\lambda$ FOR EXACT SOLUTION.
- <u>WHEN $\delta \ll 1$</u> (FINE GRID), $\sin\delta \rightarrow \delta$ AND $1 - \cos\delta \rightarrow 0$,

  THEN $\tan\varepsilon \rightarrow (e - w)\delta \equiv \frac{u\ \ t}{(x_P - x_W)} \cdot \frac{(x_P - x_W)}{\lambda} = \frac{u\ \ t}{\lambda}$,

  THEREFORE, $\varepsilon \rightarrow \tan^{-1}(u\ \ t)/\lambda)$; CORRECT FOR SMALL ε.
- <u>WHEN $\delta = \pi$</u>(COARSE GRID),

  $\varepsilon = 0$, REGARDLESS OF e AND w.
- CONCLUSIONS: • δ SHOULD BE MUCH LESS THAN 1 FOR REASONABLE ACCURACY.
  - $u\ \ t/\lambda$ SHOULD ALSO BE MUCH LESS THAN 1.
  - AS A RULE, THE PREDICTED PROPAGATION IS TOO SLOW.
  - PROPAGATION SPEED DEPENDS ON WAVELENGTH. THIS IS CALLED DISPERSION; IT CAUSES DISTORTION OF PROFILE SHAPE.

HTE 1	6	HEAT CONDUCTION. DISCUSSION:
18	15	THE CHANGE IN AMPLITUDE.

- ϕ/ϕ_0 IS THE AMPLITUDE RATIO. ACCORDING TO THE EXACT SOLUTION, $\phi/\phi_0 = \exp(-\Gamma\ \ t/(\rho\lambda^2))$; SO

  $\phi/\phi_0 \rightarrow 1 - \Gamma\ \ t/(\rho\lambda^2)$ WHEN t IS SMALL.
- FOR THE FD SOLUTION, <u>WHEN δ IS SMALL</u>:

  $$\phi/\phi_0 \rightarrow \{1 - \tfrac{1}{2}(e+w)\delta^2\}/\cos\varepsilon = \left\{1 - \frac{\Gamma\ \ t}{\rho\lambda^2} + \frac{1}{2}\frac{u\ \ t}{\lambda} \cdot \frac{x_P - x_W}{\lambda}\right\}/\cos\varepsilon,$$

  THIS SHOWS, INCORRECTLY, AN INFLUENCE OF u, ON THE RATIO.
- <u>WHEN $\delta = \pi$</u>, THE FD RELATION GIVES:

  $$\phi/\phi_0 = \{1 - 2(e+w)\}/\cos\varepsilon = \left\{1 - \frac{4\Gamma\ \ t}{\rho(x_P - x_W)^2} - \frac{2u\ \ t}{(x_P - x_W)}\right\}/\cos\varepsilon,$$
- HENCE, $|\phi/\phi_0|$ MAY EXCEED 1, RESULTING IN INSTABILITY, UNLESS $\Gamma t/(\rho(x_P - x_W)^2) < 1/2$ AND $ut/(x_P - x_W) < 1$.

HTE 1 18	7/15	HEAT CONDUCTION; DISCUSSION OF THE EXPLICIT PROCEDURE.

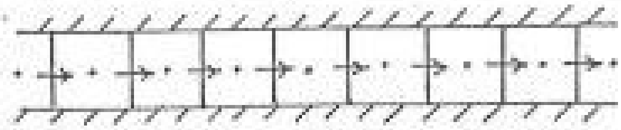
- ADVANTAGES:
  - THE PROCEDURE IS SIMPLE, BECAUSE THERE IS NO NEED TO SOLVE SIMULTANEOUS EQUATIONS.
  - IT IS ALSO EASY TO UNDERSTAND.
- DISADVANTAGES:
  - WAVE DISPERSION IS A SERIOUS DISADVANTAGE (SHARED WITH OTHER PROCEDURES ALSO).
  - THE AMPLITUDE INCREASE FOR $t \frac{\Gamma}{\rho}/(x_P-x_W)^2 > \frac{1}{2}$ OR $ut/(x_P-x_W)>1$ IS DISASTROUS, AND MUST BE AVOIDED.
  - THIS OFTEN MEANS THAT t MUST BE VERY SMALL, SO THAT COMPUTER TIMES WILL BE LONG.

HTE 1 18	8/15	HEAT CONDUCTION; GENERALISATION OF THE CONCLUSIONS ABOUT EXPLICIT PROCEDURES.

- MORE DIMENSIONS:
  - GENERALISATION TO 2 AND 3 SPACE DIMENSIONS LEAVES THE CONCLUSIONS UNCHANGED.
  - IF THE GRID SPACINGS IN THE VARIOUS DIRECTIONS DIFFER, IT IS THE SMALLER ONES WHICH GOVERN THE t LIMIT.
- NON-UNIFORM GRIDS: ALTHOUGH THE ANALYSIS IS LESS PRECISE, STILL THE SMALL-SPACING REGIONS DOMINATE THE t LIMITATION.
- OTHER EQUATIONS: WHAT IS TRUE FOR TEMPERATURE IS TRUE ALSO FOR ANY OTHER ϕ.
- VARIABLE COEFFICIENTS: EXPERIENCE SHOWS THAT THE CONCLUSIONS REMAIN TRUE.

HTE 1 18	9 15	HYDRODYNAMICS; A 2D PARABOLIC PROBLEM.

- THE PROBLEM:



SUPPOSE THAT A GAS MOVES UNSTEADILY IN A PIPE, AND THAT CONDITIONS AT EVERY CROSS-SECTION CAN BE TAKEN AS UNIFORM.

- EXPLICIT SOLUTION PROCEDURE:
  EMPLOY THE FDE, WITH $\phi = 1, u, \tilde{h}$, ETC.:
  $\phi_P = \phi_{P-} + [a + b\phi_{P-} + c_W(\phi_{W-} - \phi_{P-}) + c_E(\phi_{E-} - \phi_{P-})]/c_{P-}$
  AND EXPRESS THE CONTINUITY EQUATION ($\phi = 1$) IN THE PRESSURE-CORRECTION FORM (PANEL 11 BELOW).

HTE 1 18	10 15	HYDRODYNAMICS; EXPLICIT SOLUTION PROCEDURE FOR VELOCITY.

- MOMENTUM SOURCE: • $a + b\phi_{P-}$ REPRESENTS TWO TERMS, VIZ. THE FRICTION FORCE AND THE PRESSURE GRADIENT.
  - THE SOURCE IS EVALUATED FOR THE TIME ZERO.
  - THE p_{w-} AND p_{e-} APPEAR IN THE FDE FOR u.

- DISCUSSION: • THE LIMITATIONS OF THE SIZE OF t STILL HOLD, AS GIVEN IN PANEL 6.
  - THESE ARE $t < \frac{1}{2}(\delta x)^2\rho/\mu$ AND $t < (\delta x)/u$.
  - THE FIRST IS OPERATIVE AT LOW RE, THE SECOND AT HIGH RE.
  - OTHERWISE, NO SPECIAL PROBLEMS ARISE.

HTE 1 18	11 15	HYDRODYNAMICS; EXPLICIT FORM OF THE PRESSURE-CORRECTION EQUATION.

- ORIGIN: • PANEL 16.8 GIVES THE IMPLICIT FORM OF THE EQUATION.
  - TO DERIVE THE EXPLICIT FORM, WE REGARD THE MOMENTUM SOURCES AS BEING THOSE FOR THE BEGINNING OF THE INTERVAL; AND THE SAME IS TRUE OF ANY MASS SOURCES.
- RESULT: THE FDE BECOMES:

  $$p'_P = a/\left(\left(\frac{\partial\rho}{\partial p}\right)_P (x_e - x_w)\right)$$

  WHERE $a \equiv$ THE ERROR IN THE CONTINUITY EQUATION ASSOCIATED WITH THE u'S AND ρ'S ALREADY CALCULATED.
- NOTE: ALL p'_{i-}'s ARE ZERO BECAUSE THERE IS NO QUESTION OF CORRECTING THE PAST.

HTE 1 18	12 15	DISCUSSION OF THE EXPLICIT FORM OF PRESSURE CORRECTION EQUATION.

- USE OF THE EQUATION: p'_P IS THE INCREMENT OF PRESSURE; THUS: $p_P = p_{P*} + p'_P$.
- MAGNITUDE OF p'_P: FOR A NEARLY INCOMPRESSIBLE FLOW,

  $a \approx \rho(u_w - u_e)\ \ t$,

  HENCE, $p'_P \approx \rho(u_w - u_e)\ \ t\ /\left(\left(\frac{\partial\rho}{\partial p}\right)_P (x_e - x_w)\right)$.
- DISCUSSION: • WE HAVE $\frac{\partial\rho}{\partial p} \approx (\text{SOUND VELOCITY})^{-2}$.
  - HENCE $p'_P \approx \rho\, u^2_{sound} \cdot (u_w - u_e)\ \ t\ /(x_e - x_w)$.
  - FOR A SIMPLE VELOCITY-CHANGING WAVE, TRUE PRESSURE INCREMENT IS: $\rho u_{sound} \times$ (VELOCITY DIFFERENCE).
  - THEREFORE, FOR ACCURACY OF FD SOLUTION, $t \ll (x_e - x_w)/u_{sound}$.

HTE 1 / 18	13 / 15	FURTHER DISCUSSION OF THE EXPLICIT p' EQUATION.

- FOR INCOMPRESSIBLE FLUIDS, p'_P IS INFINITE. CONSEQUENTLY, EARLY USERS OF THIS METHOD HAD TO POSTULATE AN ARTIFICIAL COMPRESSIBILITY FOR INCOMPRESSIBLE FLUIDS, AND INDEED FOR ALL FOR WHICH $u << u_{sound}$.
- FOR 2 OR 3 SPACE DIMENSIONS, THE SITUATION IS THE SAME.
- FINALLY, IT WAS RECOGNISED THAT, EVEN IF THE EXPLICIT PROCEDURE IS TO BE USED FOR u, $\tilde{h}$, ETC., AN IMPLICIT FORMULA FOR PRESSURE CORRECTION IS NEEDED.

HTE 1 / 18	14 / 15	THE IMPLICIT EQUATION FOR PRESSURE CORRECTION, 1 SPACE AND 1 TIME DIMENSION.

- ORIGIN: FROM PANEL 16.8:
- FDE: $p'_P = (a+c_W p'_W + c_E p'_E)/\left[-b+c_W+c_E+ \frac{\partial\rho}{\partial p}(x_e - x_w)\right]$.
- COEFFICIENTS: c_W AND c_E ARE OBTAINED FROM DIFFERENTIATION OF THE FDE'S FOR u_e AND u_w, WITH APPROPRIATE ALLOWANCE FOR ρ'S.
- SOLUTION PROCEDURE: • EQUATIONS FOR ALL THE POINTS IN THE CHAIN CAN BE SOLVED SIMULTANEOUSLY BY THE TRI-DIAGONAL MATRIX ALGORITHM (SEE LATER).
  - EVEN NOW AN ELEMENT OF EXPLICITNESS CAN BE RECOVERED BY PUTTING p'_W AND $p'_E = 0$ WHEN p'_P IS TO BE CALCULATED.
- DISCUSSION: THERE IS NOW NO TIME-STEP LIMITATION.

HTE 1 18	15/15	FINAL REMARKS

- EXPLICIT PROCEDURES STILL HAVE SOME USES, BOTH FOR HEAT CONDUCTION AND HYDRODYNAMICS (SCHMIDT METHOD; MAC METHOD).
- THEIR ADVANTAGES ARE THOSE OF PRACTICAL AND CONCEPTUAL SIMPLICITY.
- THEY SUFFER FROM SEVERE TIME-STEP LIMITATIONS, BOTH AT HIGH AND LOW RE.
- IMPLICIT PROCEDURES ARE ALMOST INVARIABLY USED FOR THE PRESSURE-CORRECTION EQUATION.

HTE 1 19	1/15	LECTURE 19. IMPLICIT AND POINT BY POINT PROCEDURES

- CONTENTS:
  - GENERAL FEATURES.
  - WAVE DISPERSION AND AMPLITUDE DECAY.
  - THE GAUSS-SEIDEL PROCEDURE.
  - CONVERGENCE, UNDER- AND OVER-RELAXATION.
  - THE SIVA PROCEDURE FOR THE COUPLED HYDRODYNAMIC EQUATIONS.
- NOTES: • FULLY IMPLICIT PROCEDURES APPLY BOTH TO STEADY AND UNSTEADY PROCESSES.
  - IN THE FORMER, $t = \infty$.

HTE 1 19	2/15	GENERAL FEATURES OF IMPLICIT PROCEDURES.

- FDE: AS FOR PANEL 15.14.
- SIMULTANEITY: ϕ_P, ϕ_W, ϕ_E, ... ϕ_L, ARE ALL UNKNOWN; THE EQUATIONS FOR EACH MUST BE SOLVED SIMULTANEOUSLY.
- THERE ARE THREE MAIN METHODS:
  - POINT-BY-POINT ADJUSTMENT.
  - LINE-BY-LINE ADJUSTMENT.
  - INVERSION OF LARGER MATRICES.
- NON-LINEARITY: • DEPENDENCE OF COEFFICIENTS, ETC., ON ϕ'S NECESSITATES ITERATION.
  - THEREFORE THE LARGE-MATRIX-INVERSION METHODS ARE SELDOM USED.

HTE 1 19	3/15	HEAT CONDUCTION; A 2D PARABOLIC PROBLEM.

- PURPOSE OF THE ANALYSIS: TO DETERMINE WHETHER THE FULLY-IMPLICIT PROCEDURE IS SATISFACTORY IN RESPECT OF WAVE DISPERSION AND AMPLITUDE DECAY.
- THE PROBLEM: AS FOR PANEL 18.3.
- METHOD OF ANALYSIS: AS BEFORE, NAMELY:
  - CONSIDER THE PROPAGATION AND DECAY OF A TRAIN OF WAVES, FOR WHICH THE EXACT SOLUTION IS KNOWN.
  - COMPARE THE SOLUTION OF THE FDE'S WITH THE EXACT SOLUTION OF THE PARTIAL DIFFERENTIAL EQUATION.
  - NOTE: IT IS IMPLIED THAT THERE ARE MANY ITERATIONS AT A TIME STEP, SO THAT THE SOLUTION OF THE IMPLICIT EQUATION IS ATTAINED.

HTE 1 19	4/15	HEAT CONDUCTION; SOLUTION OF THE FULLY-IMPLICIT FDE'S FOR THE 2D PARABOLIC PROBLEM.

- FDE: $\phi_P = (\phi_{P-} + w\phi_W + e\phi_E)/(1 + w + e)$, WITH NOTATION AS FOR LECTURE 18.
- CONDITIONS AT BEGINNING OF TIME INTERVAL (TIME=0):
  $\phi_{P-} = \Phi\text{-} \sin \xi$, WITH $\xi \equiv x/\lambda$.
- CONDITIONS AT END OF TIME INTERVAL (TIME=t):
  $\phi_P = \Phi \sin(\xi + \varepsilon)$, $\phi_W = \Phi \sin(\xi + \varepsilon - \delta)$,
  $\phi_E = \Phi \sin(\xi + \varepsilon + \delta)$, WITH $\varepsilon = \frac{x\ \text{shift}}{\lambda}$, $\delta \equiv \frac{x_P - x_W}{\lambda}$
- MANIPULATIONS:
  - SUBSTITUTE INTO FDE.
  - SET $\xi = 0$ AND $\pi/2$; HENCE OBTAIN TWO EQUATIONS, FOR ε AND Φ/Φ_o.

HTE 1 19	5 — 15	WAVE DISPERSION IN THE FULLY-IMPLICIT HEAT-CONDUCTION PROBLEM.

- EQUATION FOR ε: $\tan\varepsilon = \frac{(e-w)\sin\delta}{1+(e+w)(1-\cos\delta)}$.
- DISCUSSION: • THIS DIFFERS FROM THE EXPRESSION FOR THE FULLY-EXPLICIT METHOD, IN HAVING $1+\ldots$ RATHER THAN $1-\ldots$ IN THE DENOMINATOR (SEE PANEL 18.4).
  - THE RATIO OF FD PROPAGATION SPEED TO THE EXACT-SOLUTION SPEED IS $\frac{\varepsilon}{(e-w)\delta}$, I.E., $\frac{\tan^{-1}\{\frac{(e-w)\sin\delta}{1+(e+w)(1-\cos\delta)}\}}{(e-w)\delta}$.
  - IT CAN BE SHOWN THAT THE CRANK-NICHOLSON FORMULA GIVES A RATIO INTERMEDIATE BETWEEN THOSE OF THE FULLY-IMPLICIT AND FULLY-EXPLICIT PROCEDURES. THIS TENDS TO 1.0 AS $\delta \to 0$ (LONG WAVES, I.E. $\lambda \gg x_p - x_w$): FOR THEN $\sin\delta \to \delta$ AND $\cos\delta \to 1.0$.

HTE 1 19	6 — 15	AMPLITUDE RATIO IN THE FULLY-IMPLICIT HEAT-CONDUCTION PROBLEM.

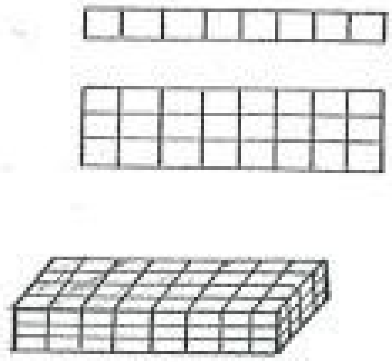
- EQUATION: $\frac{\phi}{\Phi} = \frac{\cos\varepsilon}{1+(e+w)(1-\cos\delta)}$.
- "EXPLICIT" EXPRESSION FOR COMPARISON: $\frac{1-(e+w)(1-\cos\delta)}{\cos\varepsilon}$.
- WHEN δ AND ε ARE SMALL: $\phi/\phi_{-} \to 1 - (e+w)\frac{\delta^2}{2} = 1 - \Gamma t/(\rho\lambda^2)$,

  WHICH AGREES WITH THE EXACT SOLUTION.
- WHEN $\delta = \pi$ & $\cos\delta = -1$, $\phi/\phi_0 = \cos\varepsilon/\{1+2(e+w)\}$.
  - THIS IS ALWAYS < 1, BECAUSE NEITHER e NOR w CAN BE NEGATIVE.
  - INSTABILITY THEREFORE CANNOT OCCUR; BUT THE "DAMPING" OF WAVES MAY BE SEVERE.

HTE 1 19	7 15	GENERALISATION OF CONCLUSIONS ABOUT ACCURACY OF FULLY-IMPLICIT FDE'S.

- MAIN CONCLUSIONS: • FULLY-IMPLICIT FDE'S ARE "SAFE"; THEY CANNOT LEAD TO DISASTROUS WAVE AMPLIFICATIONS, NO MATTER HOW LARGE IS t.
  - SINCE SMALL-WAVE-LENGTH WAVES (LARGE δ) ARE LIKELY TO TRAVEL AT DIFFERENT SPEEDS, AND BE MORE DAMPED, THAN LARGE-WAVE-LENGTH ONES, NON-SINUSOIDAL WAVES WILL CHANGE SHAPE AS THE INTEGRATION PROCEEDS.
- GENERALISATION: THESE CONCLUSIONS APPLY TO:
  - OTHER VARIABLES THAN TEMPERATURE,
  - 2 AND 3 SPACE VARIABLES,
  - NON-UNIFORM GRIDS,
  - ϕ-DEPENDENT COEFFICIENTS.

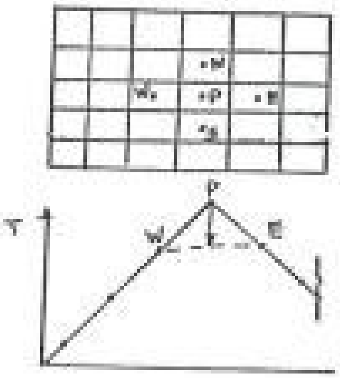
HTE 1 19	8 15	THE POINT-BY-POINT (GAUSS-SEIDEL) SOLUTION PROCEDURE.



- NATURE OF THE PROCEDURE:
  - IN THE 1D, 2D OR 3D CELL ARRAY, A "VISITING ORDER" IS ESTABLISHED.
  - AS EACH CELL IS "VISITED", THE ϕ_P IS BROUGHT INTO ACCORD WITH THE FDE.
  - THE CYCLE OF "VISITS" THROUGH ALL THE CELLS IS REPEATED UNTIL THE ADJUSTMENTS ARE NEGLIGIBLE IN AMOUNT.
- VARIANTS: • EACH ϕ (I.E. $\tilde{h}$, m, ETC.) MAY BE ADJUSTED WHEN THE CELL IS VISITED; OR ALL THE $\tilde{h}$'S AND THEN ALL THE m'S MAY BE ADJUSTED. • COEFFICIENTS MAY BE ADJUSTED IMMEDIATELY OR AFTER DELAY.

HTE 1 19	9/15	THE POINT-BY-POINT PROCEDURE; CONVERGENCE.

- ERROR SPREAD:
  - WHEN P IS ADJUSTED, THE ERROR THERE IS REDUCED TO ZERO.
  - HOWEVER, THE ERRORS IN THE NEIGHBOUR CELLS ARE CHANGED BY AN EQUAL TOTAL AMOUNT.
  - ADJUSTMENT THEREFORE TENDS TO SPREAD ERRORS RATHER THAN TO ELIMINATE THEM (BUT ERRORS OF OPPOSITE SIGN WILL CANCEL).



- EFFECTS OF FIXED-ϕ BOUNDARIES: FOR ADJUSTMENTS NEAR BOUNDARIES, THE NET ERROR IS DIMINISHED.
- CONVERGENCE: THIS RESULTS FROM SPREAD AND ELIMINATION OF ERROR.

HTE 1 19	10/15	THE POINT-BY-POINT PROCEDURE; UNDER- AND OVER-RELAXATION.

- DEFINITIONS:
  - LET ϕ_P^* ≡ RHS OF FDE FOR ϕ AT P.
  - LET ϕ_P BE CALCULATED FROM: $\phi_P = \phi_{Po} + \alpha(\phi_P^* - \phi_{Po})$ WHERE ϕ_{Po} IS THE VALUE OF ϕ_P IN STORE BEFORE ADJUSTMENT, AND α IS THE RELAXATION FACTOR.
- INFLUENCE OF α:
  - $\alpha < 1$ SLOWS DOWN CONVERGENCE.
  - $\alpha > 1$ (BUT <, SAY, 1.8) SPEEDS UP CONVERGENCE.
  - $\alpha < 1$ IS SOMETIMES NEEDED BECAUSE OF THE NEED TO SLOW DOWN THE COEFFICIENT CHANGES.
  - IF SO, IT IS USUALLY BETTER TO UNDER-RELAX THE COEFFICIENTS THAN THE MAIN VARIABLES.

HTE 1 19	11/15	THE POINT-BY-POINT PROCEDURE; DISCUSSION.

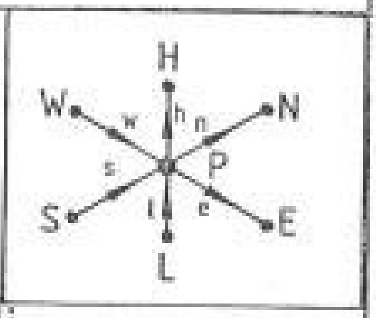
- DISTINCTION BETWEEN EXPLICIT AND POINT-BY-POINT IMPLICIT PROCEDURES:
  - THE TWO PROCEDURES LOOK SIMILAR, IN THAT FORMULAE OF THE KIND: ϕ_P = LINEAR EXPRESSION OF "KNOWN" QUANTITIES, ARE REPEATEDLY USED.
  - A "VISITING SWEEP" THUS LOOKS LIKE THE PERFORMANCE OF ONE TIME STEP IN AN EXPLICIT PROCEDURE.
  - THE EQUATIONS, AND THEIR SIGNIFICANCES, ARE HOWEVER DIFFERENT, E.G. THE IMPLICIT PROCEDURE CAN BE USED FOR A STEADY-STATE PROCESS.
- POINT-BY-POINT PROCEDURES ARE EASY TO PROGRAM, AND FAIRLY USEFUL; BUT THEY ARE SLOW FOR FINE GRIDS.

NOTE: RESULT OF A GAUSS-SEIDEL IMPLICIT SWEEP DEPENDS ON VISITING ORDER, THAT OF AN EXPLICIT JACOBI IMPLICIT SWEEP DOES NOT

HTE 1 19	12/15	POINT-BY-POINT PROCEDURE FOR THE HYDRODYNAMIC FDE'S; THE PROBLEM.

- THE FDE'S: $\phi_P = (a + c_{P-}\phi_{P-} + \sum_i c_i\phi_i)/(c_P - b)$ WITH $\phi \equiv u, v,$ w AND 1 (FOR CONTINUITY).
- THE OCCURRENCE OF PRESSURE: • PRESSURE APPEARS IN THE MOMENTUM-SOURCE TERMS, BUT HAS NO EQUATION OF ITS OWN.
  - PRESSURE DOES NOT APPEAR IN THE FOURTH EQUATION, THAT OF CONTINUITY.
- THE PROBLEM: • WE HAVE FDE'S FOR POINT-BY-POINT ADJUSTMENT OF u, v, AND w (WITH KNOWN p'S).
  - WE NEED ADJUSTMENT FORMULAE FOR THE p'S.
  - HOW WILL THE u, v, w'S SATISFY CONTINUITY?

HTE 1 19	13/15	THE SIVA (SIMULTANEOUS-VARIABLE-ADJUSTMENT) PROCEDURE.



- NATURE: THE FOLLOWING SEVEN VARIABLES ARE ADJUSTED SIMULTANEOUSLY:
  $p_P,\ u_e,\ u_w,\ v_s,\ v_n,\ w_h,\ w_e.$
- EQUATIONS USED: • THE FDE'S FOR $u_e, \ldots, w_l$ ARE WRITTEN IN THE FORM:

  $u_e = L(p_P, u_w)$ $\quad u_w = L(p_P, u_e)$

  $v_s = L(p_P, v_n)$ $\quad v_n = L(p_P, v_s)$

  $w_l = L(p_P, w_h)$ $\quad w_h = L(p_P, w_l)$

  WHERE $L(\ldots) \equiv$ LINEAR EXPRESSION OF ....
- THE CONTINUITY EQUATION IS WRITTEN AS:
  $0 = L(p_P, u_e, u_w, v_s, v_n, w_h, w_l).$

HTE 1 19	14/15	THE SIVA PROCEDURE; DISCUSSION.

- ELIMINATION: • SUBSTITUTIONS FROM THE FIRST SIX EQUATIONS INTO THE SEVENTH YIELD $p_P = L(p_N, p_S, \ldots)$ ETC., ALL p's BEING PRESUMED KNOWN, I.E. p_P = "CONST".
  - THEREAFTER ALL THE u's, v's AND w's ARE OBTAINABLE.
- USE IN ADJUSTMENT SWEEPS: SIVA DIFFERS FROM THE POINT-BY-POINT PROCEDURE FOR TEMPERATURE (SAY), ONLY IN THAT SEVEN VARIABLES ARE ADJUSTED AT EACH VISIT TO A POINT.
- CONVERGENCE: • THIS IS USUALLY OBTAINED.
  - SOMETIMES UNDER-RELAXATION IS NEEDED BECAUSE OF THE NON-LINEAR EFFECTS.

HTE 1 19	15 15	FINAL REMARKS

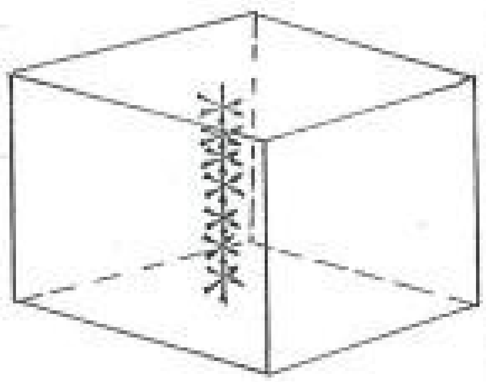
- THE IMPLICIT FDE'S ARE MORE SATISFACTORY THAN THE EXPLICIT ONES, IN THAT THEY PERMIT EVEN INFINITE t TO BE HANDLED WITHOUT INSTABILITY.
- POINT-BY-POINT PROCEDURES NORMALLY CONVERGE. CONVERGENCE MAY BE ACCELERATED BY OVER-RELAXATION; BUT UNDER-RELAXATION MAY BE NEEDED FOR CONVERGENCE WHEN THERE ARE STRONG NON-LINEARITIES.
- THE STRONG COUPLING OF THE HYDRODYNAMIC EQUATIONS NECESSITATES A SPECIAL PROCEDURE, SIVA.

HTE 1 20	1/15	LECTURE 20. LINE-BY-LINE, PLANE-BY-PLANE AND WHOLE-FIELD IMPLICIT PROCEDURES

- CONTENTS
  - LINE-BY-LINE PROCEDURES
  - THE ADI AND TDMA
  - UNDER- AND OVER-RELAXATION
  - PLANE-BY-PLANE PROCEDURES
  - WHOLE-FIELD PROCEDURES
  - WHOLE-FIELD VERSION OF SIVA
  - SIMPLE

HTE 1 20	2/15	GENERAL FEATURES OF LINE-BY-LINE IMPLICIT PROCEDURES; ADI (= ALTERNATING DIRECTION IMPLICIT).

- WHAT IS MEANT BY A LINE: A CONTINUOUS CHAIN OF GRID POINTS, ALONG WHICH THE ϕ VALUES ARE TO BE SIMULTANEOUSLY ADJUSTED.
- "VISITING" ORDER: ADJUSTMENTS CAN BE MADE TO ϕ VALUES ON GRID-POINT LINES, THESE LINES BEING VISITED IN ACCORDANCE WITH A PRE-ARRANGED ORDER.
- ALTERNATION OF LINE DIRECTION: • THE SAME POINT CAN APPEAR ON DIFFERENTLY ORIENTED LINES.
  - THE ADJUSTMENT PROCESS MAY "SWEEP" IN VARIOUS DIRECTIONS.



HTE 1 20	3 — 15	LINE-BY-LINE PROCEDURES; THE EQUATIONS SOLVED IN ONE ADJUSTMENT "TRAVERSE".

- EXAMPLE: LET THE LINE BE ORIENTED NORTH-SOUTH.
- TWO POSSIBLE FORMS OF THE EQUATIONS:

(1) $$\phi_P = \frac{c_S\phi_S}{c_P-b} + \frac{c_N\phi_N}{c_P-b} + \left[\frac{a+c_W\phi_W+\dots c_L\phi_L}{c_P - b}\right]$$

(2) $$\phi_P = \frac{c_S\phi_S+c_N\phi_N}{c_P-b-c_W\dots c_L} + \left[\frac{a+c_W(\phi_W-\phi_{P*})+\dots c_L(\phi_L-\phi_{P*})}{c_P - b - \dots c_L}\right]$$

- NOTE:
  - IN (1) ALL NEIGHBOURS TO EAST AND WEST, AND HIGH AND LOW, ARE TREATED AS KNOWN, DURING SOLUTION ALONG LINE.
  - IN (2) IT IS THE FLUXES TO THESE NEIGHBOURS THAT ARE REGARDED AS KNOWN.
  - METHOD (1) IS USUAL IN HTS METHODS, METHOD (2) ELSEWHERE.
  - ϕ_{P*} IS THE IN-STORE VALUE OF ϕ_P.

HTE 1 20	4 — 15	LINE-BY-LINE PROCEDURES; THE TDMA (≡ TRI-DIAGONAL-MATRIX ALGORITHM)

- EQUATIONS TO BE SOLVED: $\boxed{c_P'\phi = c_S\phi_{j-1} + c_N\phi_{j+1} + a'}$

  WHERE $\phi \equiv \phi_{i,j,k}$, BUT SUBSCRIPTS APPEAR ONLY WHEN $\neq i,j,k$.

  N.B. $j = 1, j_{max}$; $c_S = 0$ FOR $j = 1$; $c_N = 0$ FOR $j = j_{max}$.
- POSTULATE EQUATIONS: $\boxed{\phi = N\phi_{j+1} + A.}$
- SUBSTITUTE FOR ϕ AND ϕ_{j-1}, ABOVE, AND DEDUCE:

  $N \equiv c_N/D$, $D \equiv c_P' - c_S N_{j-1}$, $A \equiv (c_S A_{j-1} + a')/D$.
- COMPUTE D, N, A FOR $j = 1, j_{max}$ BY SUCCESSION.
- COMPUTE ϕ FOR $j = j_{max}, 1$ BY SUCCESSION

  N.B. $N = 0$ AT $j = j_{max}$ SO ϕ_{jmax+1} IS IMMATERIAL.
- RESULT: ALL ϕ'S ARE COMPUTED ALONG THE LINE; BUT (GUESSED) OFF-LINE ϕ'S EXERT INFLUENCE VIA a'.

HTE 1 20	5/15	LINE-BY-LINE PROCEDURES; DISCUSSION.

- ADJUSTMENTS ALONG A LINE REDUCE THE ERRORS TO ZERO ALONG THAT LINE; BUT ERRORS ARE, IN GENERAL, THEREBY CREATED ALONG NEIGHBOURING LINES.
- SINCE LINES USUALLY CONNECT WITH BOUNDARIES, THERE IS USUALLY SOME DIRECT ELIMINATION OF ERROR ALSO.
- OVER-RELAXATION CAN BE INCORPORATED BY USE OF THE EQUATIONS:

  $\phi = \phi_{old} + \alpha(\phi_{new} - \phi_{old})$

  WHERE $\alpha > 1$, E.G. 1.8.
- EXCESSIVE α (E.G.>2.0) MAY INCREASE ERRORS, I.E. CAUSE DIVERGENCE

HTE 1 20	6/15	LINE-BY-LINE PROCEDURES; DISCUSSION (CONTINUED).

- LINE-BY-LINE PROCEDURES ARE NORMALLY MUCH FASTER THAN POINT-BY-POINT ONES, ESPECIALLY WHEN THERE ARE MANY POINTS IN THE GRID.
- THE PROCEDURES COME IN MANY FORMS BECAUSE OF THE NUMEROUS POSSIBILITIES OF VARYING:-
  - THE DIRECTIONS OF TRAVERSES AND SWEEPS;
  - THE SEQUENCES IN WHICH THE VARIOUS ϕ'S ARE ATTENDED TO;
  - THE FREQUENCY WITH WHICH THE (NON-LINEAR) COEFFICIENTS ARE UP-DATED.
- HOWEVER, PLANE-BY-PLANE AND WHOLE-FIELD METHODS ARE FASTER STILL

HTE 1 20	7/15	PLANE-BY-PLANE PROCEDURES; A GENERALISATION OF THE TDMA

- EQUATIONS TO BE SOLVED: $c_P'\phi = c_S\phi_{j-1} + c_N\phi_{j+1} + c_W\phi_{i-1} + c_E\phi_{i+1} + a'$
  N.B. c_L, c_H, ϕ_{k-1}, ϕ_{k+1} NOW APPEAR IN a'.
- POSTULATE EQUATIONS: $\phi = N\phi_{j+1} + E\phi_{i+1} + A$
- SUBSTITUTE FOR ϕ, ϕ_{j-1}, ϕ_{i-1} ABOVE, AND DEDUCE :
  $N \equiv c_N/D$, $E \equiv c_E/D$, $D \equiv c_P' - c_S N_{j-1} - c_W E_{i-1}$,
  $A \equiv \{a' + c_W(N_{i-1}\phi_{i-1;j+1} + A_{i-1}) + c_S(E_{j-1}\phi_{i+1,j-1} + A_{j-1})\}/D$
- COMPUTE D, N, E, A for $j = 1, j_{max}$ and $i = 1, i_{max}$
- COMPUTE ϕ for $j = j_{max}, 1$ and $i = i_{max}, 1$
- RESULT: ALL ϕ's FOR PLANE
- NOTE: ON-PLANE ϕ's ($\phi_{i-1,j+1}$, $\phi_{i+1,j-1}$) HAVE HAD TO BE TO BE GUESSED; SO A AND ϕ MUST BE ITERATED

HTE 1 20	8/15	WHOLE-VOLUME SOLUTION PROCEDURE; A FURTHER GENERALISATION OF THE TDMA

- EQUATIONS TO BE SOLVED: $c_P' \equiv c_P - b$
  $c_P'\phi = c_S\phi_{j-1} + c_N\phi_{j+1} + c_W\phi_{i-1} + c_E\phi_{i+1} + c_L\phi_{k-1} + c_H\phi_{k+1} + a'$
- POSTULATE EQUATIONS: $\phi = N\phi_{j+1} + E\phi_{i+1} + H\phi_{1+1} + A$
- SUBSTITUTE FOR ϕ, ϕ_{j-1}, ϕ_{i-1}, ϕ_{k-1} ABOVE, AND DEDUCE:
  $N \equiv c_N/D$, $E \equiv c_E/D$, $H \equiv c_H/D$, $D \equiv c_P' - c_S N_{j-1} - c_W E_{i-1} - c_L H_{k-1}$
  $A \equiv \{a + c_W(N_{i-1}\phi_{i-1,j+1} + H_{i-1}\phi_{i-1,k+1} + A_{i-1})$
  $+ c_S(E_{j-1}\phi_{i+1,j-1} + H_{j-1}\phi_{j-1,k+1} + A_{j-1})$
  $+ c_L(N_{k-1}\phi_{j+1,k-1} + E_{k-1}\phi_{i+1,k-1} + A_{k-1})\}/D$
- COMPUTE ϕ's FROM BOXED EQUATION FOR WHOLE VOLUME. Uhin
- NOTE: ITERATION IS NEEDED FOR A's AND ϕ's, BUT NOT FOR N's, E's, H's, D's.

HTE 1 20	9 — 15	NOTES ON SOLUTION OF LINEAR-EQUATION SETS

- MORE RAPID CONVERGENCE (I.E. FEWER ITERATIONS FOR REDUCTION OF ERRORS) CAN BE PROCURED BY VARIOUS DEVICES, E.G. IMPROVING GUESSES OF ϕ's IN A's.
- SIMILAR ALGORITHMS CAN BE DEVISED FOR SETS OF COUPLED EQUATIONS, E.G.

$$L_1(\phi, \psi) = 0, \; L_2(\phi, \psi) = 0$$

BUT WITH MORE COMPUTATIONAL EXPENSE.

- SIMPLIFICATIONS ARE POSSIBLE WHEN THE COEFFICIENTS EXHIBIT RECIPROCITY, I.E. $c_{W,i+1} = c_{E,i}$, etc.
- BECAUSE THE COEFFICIENTS (c's) ARE USUALLY NOT CONSTANTS, IT MAY NOT BE WORTH PROCURING A HIGHLY ACCURATE SOLUTION OF THE LINEAR EQUATIONS BEFORE RECALCULATING c's.

HTE 1 20	10 — 15	A WHOLE-FIELD VERSION OF SIVA; NATURE.

- EQUATIONS (COMPARE PANEL 19.13):

$$u_e = L(p_P - p_E, u_w), \; u_w = L(p_P - p_W, u_e),$$
$$v_s = L(p_P - p_S, v_n), \; u_n = L(p_P - p_N, v_s),$$
$$w_l = L(p_P - p_L, w_h), \; w_h = L(p_P - p_H, w_e).$$

- ELIMINATION OF u, v, w FROM CONTINUITY YIELDS:

$$L(p_P, \; p_N, \; p_S, \; p_E, \; p_W, \; p_H, \; p_L) = 0.$$

- WHOLE-FIELD SOLUTION FOR p's PERMITS u, v, w TO BE SOLVED FOR THEREAFTER.
- BECAUSE OF (1) NEGLECT OF DISTANT u's IN u_e = ETC,
  (2) NON-LINEARITIES,

ITERATION IS REQUIRED.

HTE 1 20	11/15	WHOLE-FIELD SIVA; DISCUSSION

- IN PRINCIPLE, MORE "IMPLICITNESS" COULD BE BUILT IN IF A COUPLED-EQUATION ALGORITHM WERE USED.
- MOST COMMONLY-USED METHODS EMBODY LESS "IMPLICITNESS", IN INTERESTS OF SIMPLICITY.
- HOWEVER, MOST MODERN METHODS WILL EMPLOY A "POISSON EQUATION" FOR PRESSURE (OR PRESSURE CORRECTION) LIKE THAT OF PANEL 10, IN SOME FORM OR OTHER.
- DIFFERENT AUTHORS PROPOSE DIFFERENT METHODS FOR SOLVING THE EQUATIONS, ADI BEING COMMON.
- A POPULAR METHOD, NOT UNLIKE WHOLE-FIELD SIVA, IS "SIMPLE".

HTE 1 20	12/15	THE SIMPLE ALGORITHM; GENERAL FEATURES

- NAME: SIMPLE ≡ SEMI-IMPLICIT METHOD FOR PRESSURE-LINKED EQUATIONS.
- ORIGIN: PATANKAR & SPALDING, 1972
- NATURE:
  - u, v, w ARE SOLVED FOR IMPLICITLY BY ADI, WITH GUESSED PRESSURES IN MOMENTUM SOURCES.
  - u', v', w' ARE RELATED TO p' BY DIFFERENTIATION OF MOMENTUM EQUATIONS, NEGLECTING MINOR TERMS.
  - PRESSURES ARE CORRECTED VIA POISSON EQUATION FOR p', DERIVED FROM CONTINUITY, SOLVED BY ADI.
  - REPEATED GUESSES AND CORRECTIONS LEAD FINALLY TO SATISFACTION OF ALL EQUATIONS.

HTE 1 20	13 15	THE SIMPLE ALGORITHM; MORE COMPLETE DESCRIPTION.

- AT START OF ADJUSTMENT CYCLE, p, u, v, w, ETC., AT GRID POINTS HAVE THEIR BEST-ESTIMATE VALUES. (N.B. THESE MAY BE THE VALUES FOR THE END OF THE TIME INTERVAL IN QUESTION, WHICH MAY BE INFINITE.)
- WITH THE p'S WHICH ARE IN STORE, A CYCLE OF LINE-BY-LINE ADJUSTMENTS IS MADE, OVER THE WHOLE FIELD, FOR: u'S, v'S, w'S.
- THESE ARE THEN CALLED "STARRED VELOCITIES": u_*, v_*, w_*.
- THE CONTINUITY ERRORS FOR EACH CELL ARE NOW COMPUTED FROM THESE VELOCITIES.
- FDE'S FOR PRESSURE CORRECTION p''S ARE SET UP AS IN PANEL 16.8, BUT WITH c_E, c_E, ETC., OBTAINED DIRECTLY FROM DIFFERENTIATION OF MOMENTUM FDE'S.

HTE 1 20	14 15	THE SIMPLE ALGORITHM; MORE COMPLETE DESCRIPTION (CONTINUED).

- THE FDE'S FOR p''S ARE SOLVED BY A CYCLE OF LINE-BY-LINE ADJUSTMENTS OVER THE WHOLE FIELD.
- THE CORRESPONDING u''S, v''S, w''S AND p''S ARE ALSO APPLIED: ($u = u_* + u'$; ETC.).
- SOME PROPORTION OF THE p''S (LESS THAN 1) IS APPLIED TO THE PRESSURES ($p = p_* + \alpha p'$); I.E. THE PRESSURE CORRECTION IS "UNDER-RELAXED".
- AT THIS STAGE, IF ENOUGH ITERATIONS HAVE BEEN MADE FOR THE ERRORS IN THE p' EQUATION TO BECOME SMALL, THE ρu, ρv, ρw FIELDS SATISFY CONTINUITY.

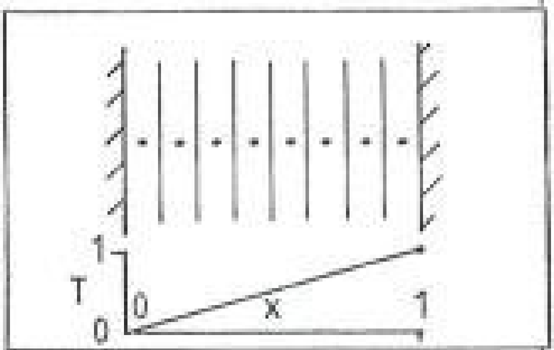
HTE 1 20	15 — 15	THE SIMPLE ALGORITHM; COMPLETION OF DESCRIPTION.

- OTHER VARIABLES ($\tilde{h}$, m_ℓ, ETC.) ARE NOW ADJUSTED BY LINE-BY-LINE SWEEPS; AND SECONDARY VARIABLES (E.G. ρ) ARE ADJUSTED CORRESPONDINGLY.
- AT THIS STAGE, ALL EQUATIONS (MOMENTUM, CONTINUITY, ENTHALPY, CHEMICAL SPECIES, ...) MAY BE OUT OF BALANCE AGAIN. THE DENSITY CHANGES ALONE COULD EFFECT THIS.
- THE CYCLE OF ADJUSTMENTS IS THEREFORE REPEATED, SUFFICIENTLY OFTEN FOR THE <u>IM</u>BALANCES TO BE REDUCED TO TOLERABLE LEVELS.
- WHEN THIS IS ACHIEVED, A "CONVERGED SOLUTION" HAS BEEN ACHIEVED; THEN THE <u>NEXT</u> TIME INTERVAL CAN BE ATTENDED TO.

HTE 1 21	1 15	PART V. PARTICULAR PROBLEMS AND PROCEDURES. LECTURE 21. HEAT CONDUCTION AND CONVECTION.

- CONTENTS:
  - HEAT CONDUCTION, POINT-BY-POINT.
  - INFLUENCES OF CELL SIZE AND ITS NON-UNIFORMITY.
  - THE USE OF THE TDMA.
  - NON-LINEARITY.
  - CONVECTION.
  - SOURCES AND SINKS.
- NOTES: • GENERAL LESSONS CAN BE LEARNED, AT LITTLE EXPENSE, BY CONTEMPLATION OF 1D EXAMPLES.

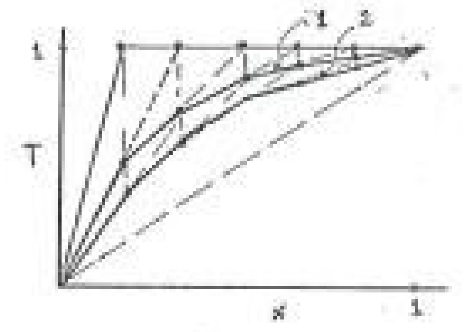
HTE 1 21	2 15	HEAT CONDUCTION; A SIMPLE PROBLEM.



- DESCRIPTION: HEAT IS CONDUCTED IN STEADY STATE THROUGH A SLAB OF UNIFORM MATERIAL.
- BOUNDARY CONDITIONS: $T=0$ AT $x=0$;
  $T=1$ AT $x=1$.
- EXACT SOLUTION: $T = x$.
- PURPOSES OF STUDY: • TO EXPLORE THE INFLUENCES ON SOLUTION TIME BY NUMERICAL MEANS OF:
  - NUMBER OF CELLS;
  - NON-UNIFORMITY OF CELL SIZE;
  - USE OF POINT-BY-POINT OR LINE-BY-LINE METHODS.
  - TO EXPLAIN RELEVANCE TO 2D AND 3D PROBLEMS.

HTE 1 21	3/15	THE SIMPLE HEAT-CONDUCTION PROBLEM; POINT-BY-POINT ADJUSTMENT (GAUSS-SEIDEL).

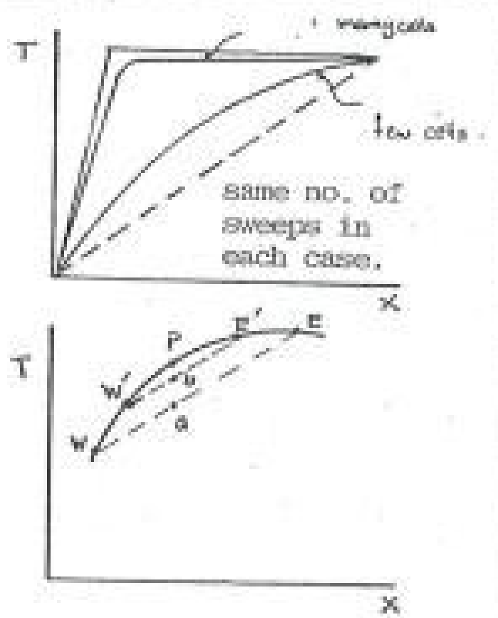
- INITIAL GUESS: T=1 AT ALL POINTS EXCEPT x=0.
- VISITING ORDER: FROM LEFT TO RIGHT (HOW WOULD RESULTS DIFFER IF THE RIGHT-TO-LEFT ORDER WERE USED?).



- RESULTS: • FIRST ADJUSTMENT SWEEP LEADS TO VALUES MARKED 1.
  - SECOND LEADS TO VALUES MARKED 2.
  - CONVERGENCE WILL OBVIOUSLY BE OBTAINED.
- DISCUSSION: • UNDER-RELAXATION WILL SLOW DOWN THE SOLUTION PROCESS. • OVER-RELAXATION ($\alpha>1$) WILL SPEED IT UP. (TRY THIS OUT.)

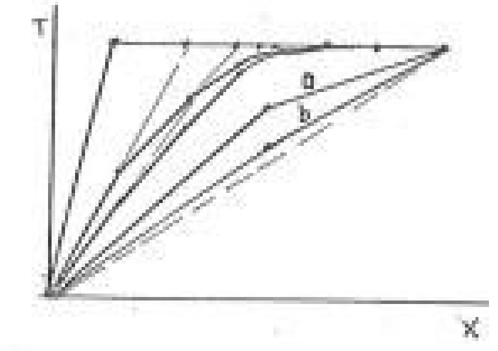
HTE 1 21	4/15	THE HEAT-CONDUCTION PROBLEM; INFLUENCE OF CELL NUMBER (SIZE).

- INFLUENCE OF CELL SIZE:
  - LOWER SKETCH SHOWS THAT MAGNITUDE OF T CHANGE $\propto \frac{d^2T}{dx^2} \cdot (x_E - x_W)^2$.
  - WITH SAME T∿x CURVE, Tp IS ADJUSTED TO a IF NEIGHBOURS ARE W AND E, TO b IF THEY ARE W' AND E'.
- SPEED OF CONVERGENCE:
  - NO. OF SWEEPS $\propto 1/(\delta x)^2$.
  - COMPUTATION PER SWEEP $\propto 1/\delta x$.
- COMPUTER TIME FOR CONVERGENCE: $\propto 1/(\delta x)^3$.



HTE 1 21	5/15	THE HEAT-CONDUCTION PROBLEM: INFLUENCE OF NON-UNIFORMITY OF CELL SIZE.

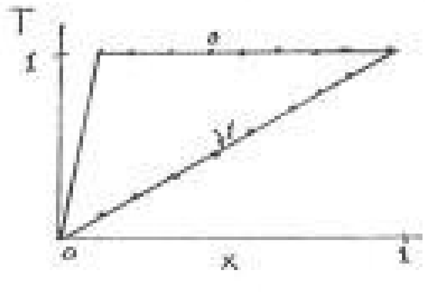
- ILLUSTRATIVE EXAMPLE:
  LET THERE BE TWO SMALL CELLS IN THE MIDDLE (1 SMALL ONE WOULD SUFFICE).



- CONSEQUENCE: • TEMPERATURE ADJUSTMENTS CAN BE LARGE EVERYWHERE BUT NEAR THE SMALL CELLS.
  - AFTER A FEW SWEEPS, THE $T \sim x$ CURVES ARE LIKE a AND b.
  - THE RATE OF CONVERGENCE IS ENTIRELY DOMINATED BY THE SIZE OF THE SMALL MIDDLE CELLS ($\propto \delta x^2_{small}$).

HTE 1 21	6/15	THE HEAT-CONDUCTION PROBLEM; APPLICATION OF THE TDMA.

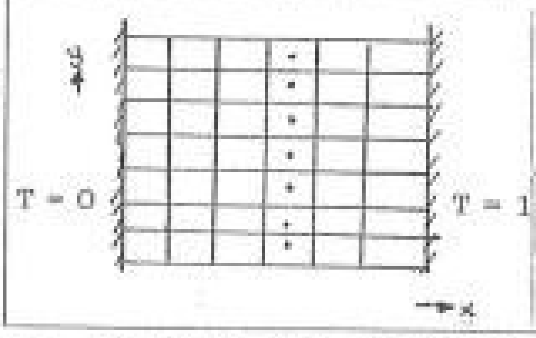
- THE PROCEDURE:
  THE EQUATIONS FOR ALL T'S ARE SOLVED SIMULTANEOUSLY BY THE TDMA ALONG THE x DIRECTION.



- THE RESULT:
  THE EXACT SOLUTION IS OBTAINED IN A SINGLE ADJUSTMENT STEP.
- INFLUENCE OF CELL NUMBER: THE COMPUTATION TIME IS DIRECTLY PROPORTIONAL TO THE NUMBER OF CELLS.
- INFLUENCE OF NON-UNIFORMITY OF CELL SIZE: THERE IS NO INFLUENCE.

HTE 1 21	7/15	RELEVANCE OF THE 1D HEAT-CONDUCTION PROBLEM TO USE OF TDMA IN A 2D PROBLEM.

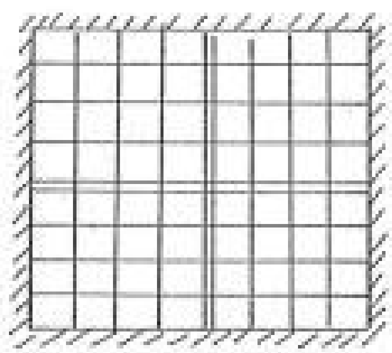
- THE PROBLEM: SUPPOSE THAT A 2D CELL STRUCTURE IS BEING USED, EVEN THOUGH THE BC'S HAPPEN, IN THIS CASE, TO GIVE A 1D SOLUTION.



- THE METHOD: SUPPOSE THAT THE TDMA IS BEING USED, BUT IN THE y DIRECTION, SO THAT ALL T'S AT 1x ARE ADJUSTED AT ONCE.
- CONVERGENCE RATE: • THIS WILL BE THE SAME AS FOR POINT-BY-POINT ADJUSTMENT FOR THE 1D CELL CHAIN. • IT WILL OF COURSE BE MUCH FASTER THAN POINT-BY-POINT ADJUSTMENT FOR THE 2D CELL ARRAY. • THE CONVERGENCE RATE WILL BE $\alpha(\delta x)^2$ AND SO WILL DIMINISH WITH INCREASING CELL NUMBER. • A SINGLE ROW OF CELLS WITH SMALL δx WILL DECELERATE CONVERGENCE.

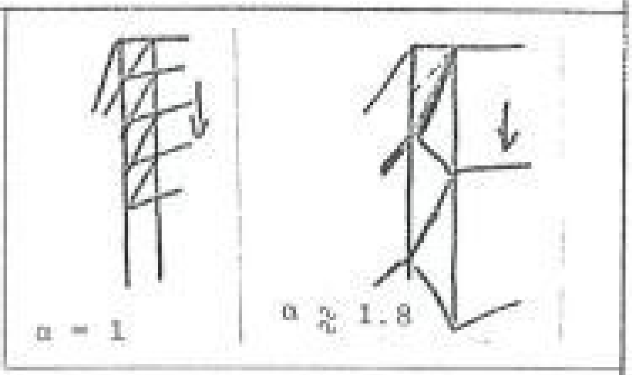
HTE 1 21	8/15	HEAT CONDUCTION; A DIFFICULT 2D PROBLEM.

- AN OBVIOUS SOLUTION TO THE PANEL 7 PROBLEM: THE TDMA TRAVERSES SHOULD BE MADE IN THE x DIRECTION ALSO; THEN SMALL-δx CELLS ARE DEALT WITH EASILY. N.B. THE EXACT SOLUTION IS OF COURSE NO LONGER OBTAINED <u>IMMEDIATELY</u>.
- A CELL ARRANGEMENT WHICH WILL STILL MAKE FOR SLOW CONVERGENCE:
  - SUPPOSE THAT THERE ARE TWO THIN-CELL STRIPS.
  - THE ALTERNATION OF TRAVERSE DIRECTIONS CANNOT ACCELERATE CONVERGENCE MUCH.



HTE 1 21	9/15	A PARTIAL SOLUTION OF THE THIN-CELL PROBLEM.

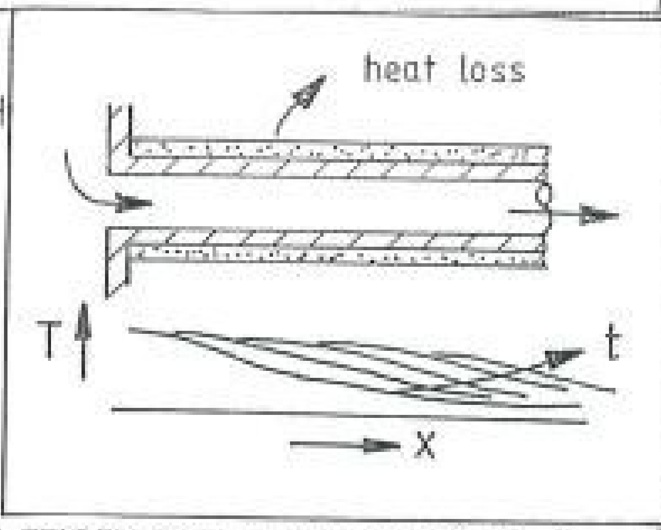
- NATURE: USE OVER-RELAXATION FOR CELLS RESPONSIBLE FOR SLOW CONVERGENCE ($\alpha < 2$).
- RESULT: • TEMPERATURE ADJUSTMENTS AT FIRST INCREASE IN SIZE AS THE ADJUSTMENTS PROCEED.
  - CONVERGENCE IS MUCH MORE RAPID.



- SUGGESTION: TRY THIS, USING GRAPHICAL MEANS.
- PRACTICAL RELEVANCE: TO 2D AND 3D LINE-BY-LINE PROCEDURES AS WELL AS 1D POINT-BY-POINT PROCEDURES.

HTE 1 21	10/15	A PROBLEM WITH COMBINED CONDUCTION AND CONVECTION; 1D UNSTEADY

- PHYSICAL DESCRIPTION:
  - WARM WATER FLOWS THROUGH A METAL PIPE, PROTECTED BY INSULATION FROM A COOLER ENVIRONMENT.
  - THE INSULATION IS IMPERFECT.
  - TEMPERATURES WITHIN THE WATER AND METAL DEPEND ONLY ON x AND t.



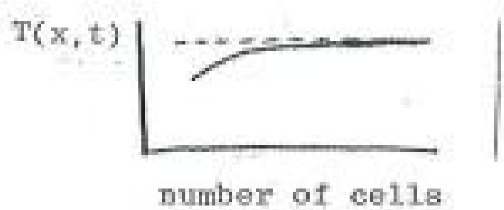
- THE PROBLEM: CALCULATE THE TEMPERATURE DISTRIBUTION AS A FUNCTION OF TIME, AFTER THE WARM-WATER SUPPLY IS SUDDENLY STARTED.

HTE 1 21	11 15	THE CONDUCTION-CONVECTION PROBLEM; POINT-BY-POINT SOLUTION PROCEDURE.

- NATURE OF THE PROCEDURE: • DETERMINE COEFFICIENTS IN FDE, TAKING ACCOUNT OF x-WISE CONVECTION AND HEAT LOSS (NEGATIVE SOURCE) THROUGH THE INSULATION.
  - ESTABLISH "VISITING ORDER", AND TIME-STEP SIZE.
  - MAKE POINT-BY-POINT VALUE-ADJUSTMENT SWEEPS, ITERATING FOR EACH TIME STEP UNTIL ERRORS, IN TERMS OF UNBALANCED HEAT SOURCES, ARE SMALL. (N.B. SMALL ADJUSTMENTS ARE NOT A SIGN OF CONVERGENCE. CONSIDER THE SMALL-CELL PROBLEM.)
  - PROCEED THUS FROM TIME STEP TO TIME STEP.
- RESULTS: A CONVERGED SOLUTION + A COMPUTER BILL.

HTE 1 21	12 15	THE CONDUCTION-CONVECTION PROBLEM; USE OF THE TDMA.

- NATURE OF THE PROCEDURE: AS FOR PANEL 11, EXCEPT THAT THE WHOLE ADJUSTMENT IS PERFORMED IN A SINGLE STEP.
- INFLUENCE OF NUMBER OF CELLS: • COMPUTER TIME PROPORTIONAL TO NUMBER OF CELLS.
  - ACCURACY INCREASES WITH NUMBER OF CELLS; BUT, OF COURSE, BEYOND A SUFFICIENT NUMBER, NO WORTHWHILE IMPROVEMENT IS ACHIEVED.



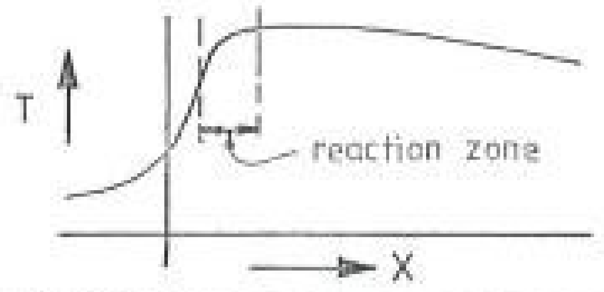
- INFLUENCE OF NUMBER OF TIME STEPS: • COMPUTER TIME INCREASES WITH $1/\delta t$.
  - ACCURACY ALSO INCREASES.

HTE 1 21	13/15	THE HEAT-CONDUCTION PROBLEM; NON-LINEAR EFFECTS.

- HOW NON-LINEARITY MAY ARISE IN PRACTICE: • THE HEAT-LOSS-TO-SURROUNDINGS LAW MAY BE NON-LINEAR (E.G. FREE CONVECTION, RADIATION).
  - THE METAL CONDUCTIVITY MAY DEPEND UPON TEMPERATURE.
- HOW NON-LINEARITY EXPRESSES ITSELF:
  - THE FINITE-DIFFERENCE COEFFICIENTS DEPEND UPON THE NEARBY TEMPERATURES.
  - BEGINNING-OF-INTERVAL OR LATEST-IN-STORE VALUES OF T MAY BE USED.
- CONSEQUENCES: • EVEN FOR STEADY STATE, AND WITH USE OF TDMA, ITERATION IS NEEDED.
  - IN UNSTEADY-STATE PROCESSES, WITH SMALL δt, ITERATION MAY NOT BE NEEDED.

HTE 1 21	14/15	A 1D PROBLEM WITH HEAT CONDUCTION, CONVECTION AND TEMPERATURE-DEPENDENT CHEMICAL REACTION; FLAME PROPAGATION.

- THE PROCESS: PRE-MIXED FUEL + AIR FLOWS STEADILY THROUGH A COOLED POROUS PLUG, AND BURNS ON THE DOWNSTREAM SIDE.



- PROBLEM: DETERMINE FLOW RATE WHICH JUST CAUSES THE FLAME TO DETACH ITSELF ($dT/dx = 0$) AT THE PLUG SURFACE.
- METHOD OF ANALYSIS: • SET UP FINITE-DIFFERENCE EQUATIONS (USING GUESSED T'S TO GIVE r'S, REACTION RATES).
  - SOLVE BY TDMA. • VARY FLOW RATE UNTIL CONDITION ($dT/dx=0$) PLUG IS ACHIEVED.

HTE 1 21	15/15	FINAL REMARKS

- EXAMINATION OF 1D PROBLEMS PERMITS MANY GENERALLY-APPLICABLE LESSONS TO BE LEARNED (E.G. SUPERIORITY OF LINE-BY-LINE METHODS; THE DAMAGING EFFECTS OF HAVING EVEN ONE SMALL CELL; HOW OVER-RELAXATION MAY HELP).
- NON-LINEAR EFFECTS NORMALLY REQUIRE ITERATION, UNLESS SMALL STEPS ARE TAKEN IN TIME-DEPENDENT PROBLEMS.
- SMALL δx AND δt ARE NEEDED FOR HIGH ACCURACY; BUT THEY INCREASE THE COMPUTER EXPENSES.
- IN SOME NON-LINEAR PROBLEMS, WHERE $\lambda = \lambda\,(T)$, USE OF A NEW VARIABLE, $T'(\equiv \int(\lambda/\lambda_{ref})\,dT)$, IS ADVANTAGEOUS; COEFFICIENTS CAN BE COMPUTED ONCE FOR ALL.

HTE 1 22	1/15	LECTURE 22. FLUID FLOW THROUGH A POROUS MEDIUM.

- CONTENTS:
  - FIXED-PRESSURE-DIFFERENCE PROBLEMS.
  - THE LINEAR RESISTANCE LAW.
  - THE QUADRATIC RESISTANCE LAW.
  - LINEARISATION AND OVER-RELAXATION.
  - FIXED-FLOW PROBLEMS.
- NOTES:
  - 1D PROBLEMS ARE DISCUSSED, FOR SIMPLICITY.
  - PRACTICAL RELEVANCE IS TO FLOW IN OIL RESERVOIRS, "PACKED BEDS", HEAT-EXCHANGER SHELLS.
  - THEORETICAL IMPORTANCE LIES IN EMPHASIS GIVEN TO THE PRESSURE (OR PRESSURE-CORRECTION) EQUATIONS.

HTE 1 22	2/15	UNSTEADY FLOW OF A COMPRESSIBLE FLUID WITH A LINEAR RESISTANCE LAW (D'ARCY's LAW).

- DEFINITION: • FROM PANEL 6.11, $\rho u = - \frac{\rho}{F} \frac{\partial p}{\partial x}$.
  - (ρ / F) IS TAKEN AS INDEPENDENT OF x AND t.
- DIFFERENTIAL EQUATION EXPRESSING MASS CONSERVATION:

  $\boxed{\left(\frac{d\rho}{dp}\right) \frac{\partial p}{\partial t} - \frac{\rho}{F} \frac{\partial^2 p}{\partial x^2} = 0}$, WHERE $(d\rho / dp)$ WILL ALSO BE TAKEN AS INDEPENDENT OF x AND t. (SEE PANEL 6.12)
- ANALOGY WITH HEAT CONDUCTION: THIS OBEYS THE PARTIAL DIFFERENTIAL EQUATION:

$$c \rho \frac{\partial T}{\partial t} - \lambda \frac{\partial^2 T}{\partial x^2} = 0.$$

- CONCLUSION: ALL THAT HAS BEEN LEARNED ABOUT HEAT CONDUCTION APPLIES ALSO TO THIS PROBLEM.

HTE 1 22	3 15	LINEAR RESISTANCE WITH FIXED OVERALL PRESSURE DIFFERENCE; POINT-BY-POINT SOLUTION.

- SIMILARITY TO HEAT CONDUCTION: THERE IS NO ESSENTIAL DIFFERENCE FROM THE HEAT-CONDUCTION PROBLEM; THE POINT-BY-POINT PROCEDURE, WITH ITERATION, CAN THEREFORE BE EMPLOYED.
- SPATIALLY VARYING F/ρ: • IF F/ρ VARIES WITH x (BUT NOT WITH t OR p), THERE IS NO ESSENTIAL DIFFERENCE; THE EFFECT IS JUST LIKE THAT OF HAVING NON-UNIFORM δx.
  - IN PARTICULAR, A VALUE OF LOCALLY HIGH F/ρ (WITH UNIFORM δx) WILL HAVE THE EFFECT OF SLOWING DOWN CONVERGENCE.
- CONVERGENCE: THIS IS ASSURED; THE TASK IS TO OBTAIN IT QUICKLY, E.G. BY OVER-RELAXATION.

HTE 1 22	4 15	LINEAR RESISTANCE WITH FIXED OVERALL PRESSURE DIFFERENCE; USE OF THE TDMA.

- INFLUENCE ON COMPUTER TIME: BECAUSE THE TDMA PRODUCES THE REQUIRED EXACT SOLUTION OF THE FDE'S IN THE ADJUSTMENT (PER TIME STEP),THE COMPUTER TIME IS GREATLY REDUCED WHEN THE NUMBER OF INTERVALS IS LARGE AS COMPARED WITH PBP.
- ADVANTAGE OVER POINT-BY-POINT PROCEDURE:
  - THIS INCREASES THE LARGER IS δt.
  - FOR SMALL δt, THE POINT-BY-POINT ADJUSTMENT MAY SUFFICE BECAUSE THE $(d\rho/dp)$ TERMS DOMINATE.
  - IN THESE CIRCUMSTANCES, EVEN EXPLICIT METHODS MAY SUFFICE.

HTE 1 22	5 — 15	THE QUADRATIC RESISTANCE LAW; MATHEMATICAL SIGNIFICANCE

- DEFINITION: SUPPOSE $\rho u^2 = - K \frac{\partial p}{\partial x}$, K = const. .

  THIS IS NOT UNCOMMON IN PRACTICE.
- COMPARISON WITH HEAT CONDUCTION: DEPENDENCE OF THERMAL CONDUCTIVITY ON TEMPERATURE IS ONE KIND OF NON-LINEARITY, BUT VERY DIFFERENT FROM THIS ONE. HERE THE "CONDUCTIVITY" $(\equiv \rho u/(-\frac{\partial p}{\partial x}))$ DEPENDS ON THE "FLUX" (ρu) RATHER THAN THE "POTENTIAL" (p).
- CONSEQUENCE: A SPECIAL KIND OF OVER-RELAXATION PROVES TO BE ADVANTAGEOUS IN THIS CASE.
- IMPORTANCE: NEARLY ALL PRESSURE DIFFERENCE~FLOW-RATE REGULATIONS <u>ARE</u> NON-LINEAR.

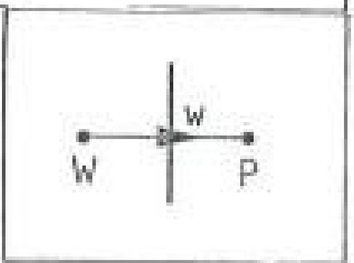
HTE 1 22	6 — 15	THE QUADRATIC RESISTANCE LAW; FIXED PRESSURE DIFFERENCE, UNSTEADY STATE.

- THE PROBLEM: $\rho u/(-\partial p/\partial x)$ IS NON-UNIFORM, AND NOT KNOWN.
- USE OF POINT-BY-POINT PROCEDURE: ITERATION NEEDED BOTH FOR "LINEAR" AND "NON-LINEAR" REASONS.
- USE OF TDMA PROCEDURE: ITERATION NEEDED "FOR NON-LINEAR REASONS" ONLY.
- BEST PROCEDURE: PROBABLY THE TDMA PROCEDURE, WITH ITERATION NUMBER DIMINISHING AS δt DIMINISHES.
- OVER-RELAXATION: THE RATE OF CONVERGENCE CAN BE IMPROVED BY PARTLY ACCOUNTING FOR THE NON-LINEARITY OF THE RESISTANCES, BY WAY OF LINEARISATION, AMOUNTING TO OVER-RELAXATION.

HTE 1 22	7/15	LINEARISING THE RESISTANCE LAW

- THE RESISTANCE LAW: $\rho u^2 = -K \frac{\partial p}{\partial x}$,

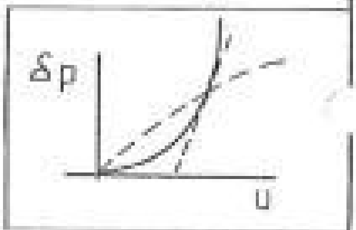
  I.E. $u_W = \left(\frac{K}{\rho} \frac{(p_W - p_P)}{x_P - x_W}\right)^{\frac{1}{2}}$.



- LINEARISED FORM:

$$u_W = u_{W*} + \frac{\partial u_W}{\partial(p_W - p_P)} \cdot (p'_W - p'_P)$$

$$= u_W^* + \frac{1}{2} \frac{u_{W*}}{p_{W*} - p_{P*}} \cdot (p'_W - p'_P)$$



- COMMENT: THIS INDICATES THAT THE PRESSURE DIFFERENCE INCREASES TWICE AS STEEPLY AS IF THE RESISTANCE WERE A LINEAR ONE.

HTE 1 22	8/15	USE OF THE LINEARISED RESISTANCE LAW.

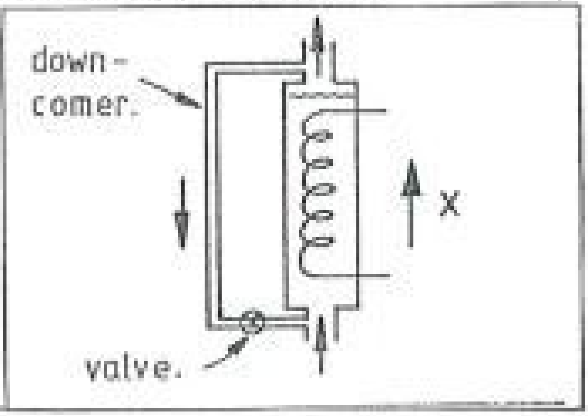
- THE PRESSURE-CORRECTION EQUATION:
  - IT IS ALWAYS POSSIBLE TO WRITE FDE'S IN THE CORRECTION FORM INSTEAD OF THE DIRECT FORM.
  - IT IS CONVENIENT TO DO SO WHEN LINEARISED LAWS ARE IN USE (AS IN SIMPLE).
  - THIS IS THEREFORE RECOMMENDED IN THE PRESENT CASE.
- THE CONSEQUENCE IS THAT THE COEFFICIENT OF p'_W, FOR EXAMPLE, IS $\frac{1}{2}\rho\, u_{W*}/(p_{W*} - p_{P*})$(OR EQUIVALENT IN TERMS OF K BUT WITHOUT $(p_{W*} - p_{P*})$), WHICH IS HALF THAT IF NO LINEARISATION HAD OCCURRED.
- THE EFFECT IS THAT OF OVER-RELAXATION OF p'.

HTE 1 22	9 15	SOME FURTHER EXAMPLES; ARBITRARY RESISTANCE LAW.

- EXAMPLES:
  - THE RESISTANCE MAY DEPEND UPON VELOCITY TO SOME OTHER POWER.
  - THE RESISTANCE MAY DEPEND UPON PRESSURE ITSELF (AS CONDUCTIVITY DEPENDS UPON TEMPERATURE) IN CONSEQUENCE OF "CRUSHING" OR "SWELLING".
- SOLUTION PROCEDURE:
  - THE VARIATIONS OF FLOW RATE WITH PRESSURES ARE STILL BEST HANDLED BY LINEARISATION.
  - NO BLIND UNDER-OR OVER-RELAXATION IS TO BE RECOMMENDED; FOR DIVERGENCE CAN THEN EASILY OCCUR.

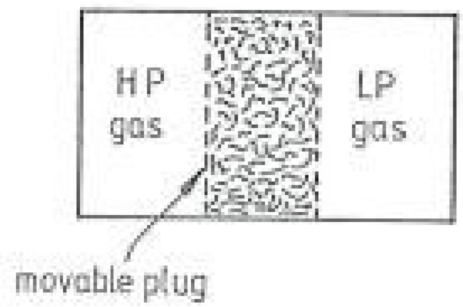
HTE 1 22	10 15	SOME FURTHER EXAMPLES; CIRCULATION PROBLEMS.

- PRACTICAL OCCURRENCE:
  - STEAM IS FORMED IN A BOILER AS THE RESULT OF CONTACT WITH TUBES CONTAINING HOT FLUID.
  - THE TUBES FORM A RESISTANCE TO THE FLOW OF STEAM, AS DOES ALSO THE THROTTLE VALVE.
  - CIRCULATION IS CAUSED BY THE DENSITY DIFFERENCE.
- PROBLEM: COMPUTE THE FLOW RATE AND STEAM CONDITION, GIVEN THE HEAT INPUT.
- PROCEDURE:
  - THE TDMA WITH LINEARISATION AND ITERATION MAY BE USED.
  - FOR STEADY STATE USE A QUADRATURE PROCEDURE.



HTE 1 22	11/15	SOME FURTHER EXAMPLES; MOVING RESISTANCES.

- PROBLEM: THE RESISTANCE CAN MOVE, EITHER PIECEWISE OR AS A BLOCK, UNDER THE INFLUENCE OF THE PRESSURE DIFFERENCE.
- SPECIAL FEATURES: THE MOMENTUM EQUATION FOR THE FLUID WILL PROBABLY TAKE THE FORM
  $\rho(u - u_{res}) = -\frac{\rho}{F}\frac{\partial p}{\partial x}$
- CALCULATION PROCEDURES:
  (1) u_{res} IS OBTAINED BY <u>INTERSPERSED</u> SOLUTIONS OF THE EQUATION OF MOTION OF THE RESISTANCE.
  OR (2) u_{res} IS SOLVED <u>SIMULTANEOUSLY</u>.

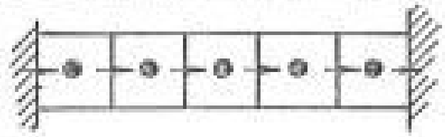


HTE 22	12/15	THE QUADRATIC RESISTANCE LAW; FIXED PRESSURE DIFFERENCE, STEADY STATE.

- THE PROBLEM: ρu IS UNIFORM, BUT NOT KNOWN.
- POSSIBLE PROCEDURES: (1) TREAT AS TRANSIENT, SOLVE EXPLICIT FDE's, PROCEED TO LARGE TIME. (2) SOLVE IMPLICIT FDE's, SET δt = LARGE, ITERATE TO CONVERGENCE.
- BEST PROCEDURE: SOLVE FOR ρu FROM $\rho u^2 = (p_1 - p_o)/\int_{x_1}^{x_2} K\,dx$. THIS REQUIRES NO ITERATION.
- CONCLUSION: IF SPEED IS IMPORTANT, THINK CAREFULLY BEFORE SELECTING A SOLUTION PROCEDURE.

HTE 1 22	13 15	LINEAR RESISTANCE PROBLEM SOLVED NUMERICALLY; FIXED FLOW RATE, DENSITY, STEADY STATE.

- SUPPOSE ρu IS FIXED AT THE START: THE QUESTION IS, WHAT PRESSURE DISTRIBUTION WILL PREVAIL IN THE MEDIUM?



- THE NATURE OF PRESSURE: • PRESSURE IS A "RELATIVE VARIABLE", I.E. A CONSTANT VALUE MAY BE ADDED TO ALL PRESSURES WITHOUT OTHERWISE ALTERING THE PROCESS (N.B. THIS IS TRUE BECAUSE $d\rho/dp = 0$ HAS BEEN ASSUMED HERE).
- SOLUTION PROCEDURE: • THERE IS NO NEED FOR ANY ITERATION; FOR PRESSURES CAN BE DETERMINED (EXCEPT FOR A CONSTANT), BY QUADRATURE:

$$p = -\frac{1}{\rho u}\int_0^x \frac{F}{\rho}\,dx \; .$$

HTE 1 22	14 15	THE QUADRATIC RESISTANCE LAW; FIXED FLOW RATE, STEADY STATE.

- THE PROBLEM: SUPPOSE THAT ρu IS FIXED AND UNIFORM.
- SOLUTION PROCEDURE: • NEITHER POINT-BY-POINT NOR LINE-BY-LINE ADJUSTMENT PROCEDURES ARE APPROPRIATE.
  - WHAT IS NEEDED IS NUMERICAL INTEGRATION OF $\frac{dp}{dx} = -\frac{\rho u^2}{K}$ (I.E. QUADRATURE $\int \frac{\rho u^2}{K}\,dx$).
  - THIS WILL LEAD TO SOLUTION IN ONE OPERATION WHEREAS BOTH POINT-BY-POINT AND LINE-BY-LINE ADJUSTMENTS WOULD NECESSITATE ITERATION.
- REMARK: BOUNDARY CONDITIONS SHOULD BE CONSIDERED BEFORE A SOLUTION PROCEDURE IS CHOSEN.

HTE 1 22	15/15	FINAL REMARKS

- POROUS-MEDIUM FLOWS ARE SIMILAR TO HEAT-CONDUCTION PROCESSES IN MANY RESPECTS.
- THE TYPICAL NATURE OF THE NON-LINEARITY IS DIFFERENT HOWEVER.
- LINEARISATION ASSISTS CONVERGENCE. THIS IS OF THE KIND USED IN SIMPLE; AND IT IS MORE SATISFACTORY FOR POROUS-MEDIUM FLOWS BECAUSE THERE IS NO MOMENTUM-CONVECTION EFFECT TO BE NEGLECTED.
- THE MARCHING-INTEGRATION, QUADRATURE, AND CIRCUIT-ANALYSIS CONCEPTS ARE USEFUL ALSO IN 2D AND 3D FLOWS WHEN 1D CORRECTIONS ARE TO BE APPLIED.

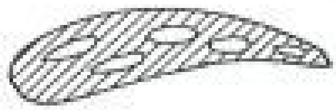
HTE 1 23	1/15	LECTURE 23. TWO-DIMENSIONAL HEAT CONDUCTION & CONVECTION.

- CONTENTS:
  - EXAMPLES.
  - FINITE-DIFFERENCE EQUATIONS.
  - POINT-BY-POINT PROCEDURES.
  - LINE-BY-LINE PROCEDURES.
  - PARABOLIC PROCEDURES.
  - NEAR-PARABOLIC PROBLEMS.
  - THE TVA.
  - THE PEA.
  - BLOCK RELAXATION.

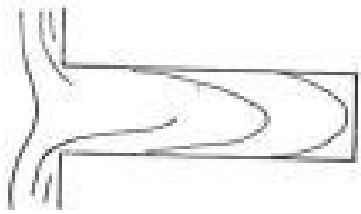
N.B. LECTURE IS RELEVANT TO ALL SINGLE-EQUATION PROBLEMS, E.G. DIFFUSION, AS WELL.

HTE 1 23	2/15	EXAMPLES; CONDUCTION.

- COOLING OF A GAS-TURBINE BLADE, STEADY OR UNSTEADY.



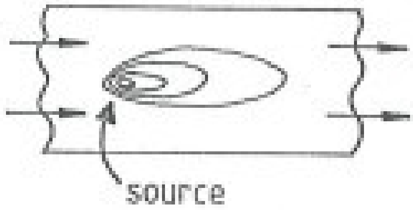
- HEAT LOSS FROM A THICK FIN.

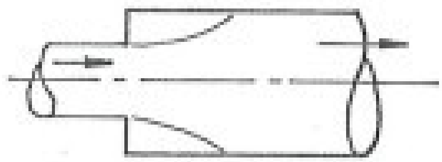


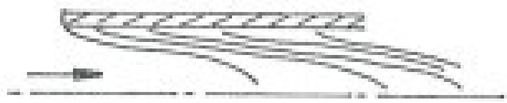
- UNSTEADY HEATING OF A CYLINDRICAL METAL INGOT.
- THERMAL STRESS IN A GLASS, PLUNGED SUDDENLY INTO HOT WATER.

HTE 1 23	3 15	EXAMPLES; CONDUCTION PLUS CONVECTION.

- STEADY-STATE CONDUCTION IN THIN MOVING STRIP, HEATED BY FIXED FLAME.
- HEAT TRANSFER FROM TURBULENT FLUID TO WALL OF PIPE ENLARGEMENT, WHEN FLOW PATTERN AND TURBULENCE FIELD ARE KNOWN.
- TRANSIENT TEMPERATURE FIELD IN FLUID RESULTING FROM SUDDEN CHANGE IN INLET TEMPERATURE.



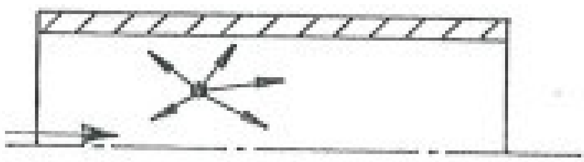


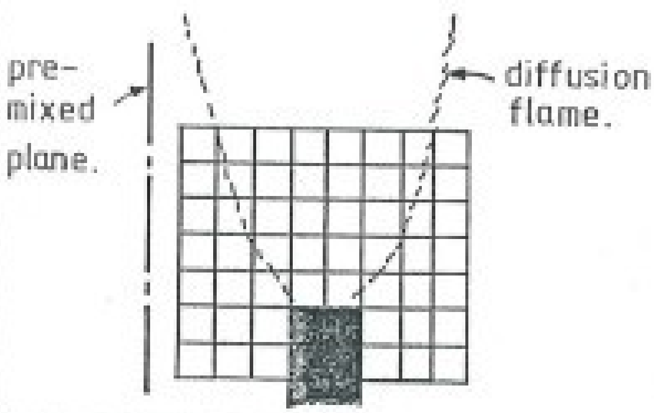


HTE 1 23	4 15	EXAMPLES; CONDUCTION PLUS CONVECTION PLUS SOURCE.

- HEAT LOSS BY CONDUCTION AND RADIATION ("THIN GAS") TO WALL OF DUCT CONTAINING STEADILY-FLOWING FLUID.
- ANCHORING OF FLAME AT BUNSEN-BURNER LIP, WHEN HYDRODYNAMICS IS PRESUMED.

N.B. HEAT CONDUCTION, DIFFUSION, REACTION, INTERACT.



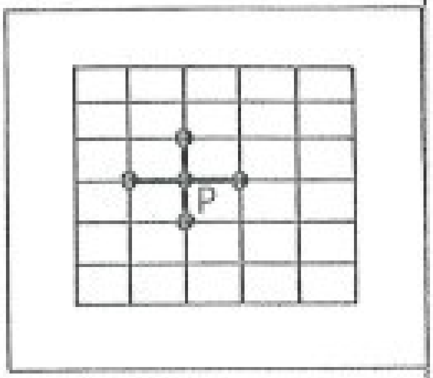


HTE 1 23	5 15	THE PROBLEM TO BE SOLVED

(Note change of symbols)

- FDE'S: • DIRECT FORM: $\phi_p = \Sigma a_n \phi_n + b$
  - CORRECTION FORM: $\phi'_p = \Sigma a_n \phi'_n + \left[\Sigma a_n \phi_{n*} + b - \phi_{p*}\right]$

    I.E. $\phi'_p = \Sigma a_n \phi'_n + c$.
- GRID: EACH P HAS FOUR NEIGHBOURS (n'S) IN SPACE, 1(P-) IN TIME.



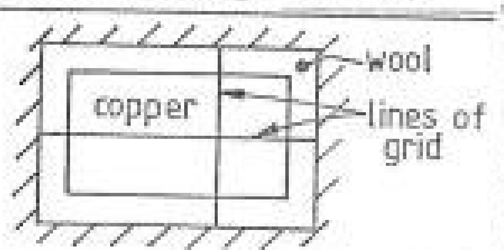
- COEFFICIENTS MAY BE:
  - MUCH LARGER/SMALLER FOR N-S THAN E - W DIRECTIONS.
  - MUCH LARGER/SMALLER FOR E THAN W, OR N THAN S SIDES.
  - DEPENDENT/INDEPENDENT OF ϕ.
  - UNIFORM/VARYING OVER FIELD.

HTE 1 23	6 15	THE POINT-BY-POINT SOLUTION PROCEDURE; REMINDER (LECTURE 19).

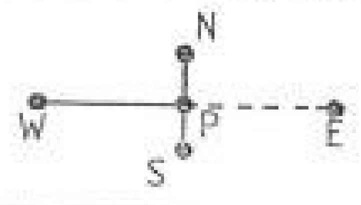
- MAIN FEATURES: • VISITING ORDER. • UNDER/OVER-RELAXATION. • UPDATING OF COEFFICIENTS WHEN NEEDED. • DIFFUSION OF ERRORS TO BOUNDARIES. • CONTINUATION UNTIL ERRORS ARE SUFFICIENTLY DIMINISHED. • TIME STEPS TAKEN IN SEQUENCE IN TRANSIENT PROBLEMS.
- ADVANTAGES: • SIMPLICITY OF PROGRAMMING AND OF CONCEPT. • VISITING ORDER MAY BE VARIED EASILY (E.G. THOSE POINTS WITH LARGE ERRORS MAY BE DEALT WITH FIRST). • COEFFICIENTS MAY BE UPDATED IMMEDIATELY.
- DISADVANTAGES: • SLOW DIFFUSION OF ERRORS TO BOUNDARIES ENTAILS LARGE COMPUTER TIMES WHEN POINTS ARE NUMEROUS.

HTE 1 23	7/15	THE LINE-BY-LINE SOLUTION PROCEDURE; REMINDER (LECTURE 20).

- MAIN FEATURES:
  - VISITING ORDER FOR LINES.
  - UNDER/OVER-RELAXATION.
  - UPDATING OF COEFFICIENTS.
  - TIME STEPS TAKEN IN SEQUENCE.
  - CONSTANT VALUE OR CONSTANT FLUX TO NODES FLANKING THE LINES.
- ADVANTAGES:
  - MORE RAPID DIFFUSION OF ERRORS TO BOUNDARIES.
  - HENCE REDUCED COMPUTER TIME, ESPECIALLY FOR MULTI-NODE GRIDS.
- DISADVANTAGES:
  - STILL UNABLE TO SECURE FAST CONVERGENCE WHEN LARGE AND SMALL CO-EFFICIENTS ARE AWKWARDLY COMBINED AS IN EXAMPLE SHOWN.



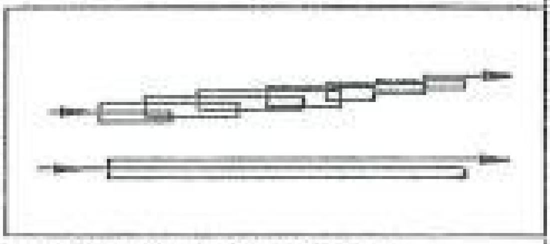
HTE 1 23	8/15	PARABOLIC PROBLEMS



- NATURE: ALL a_E'S ZERO (SAY).
- OCCURRENCE: STRONG FLOW FROM LEFT TO RIGHT (WEST TO EAST).
- SUITABLE LINE-BY-LINE SOLUTION PROCEDURE:
  - TRAVERSES ON NORTH-SOUTH LINES ONLY.
  - VISITING ORDER "SWEEPS" FROM WEST TO EAST.
- CONSEQUENCES:
  - THE EXACT SOLUTION IS OBTAINED BY A <u>SINGLE</u> SWEEP FOR LINEAR PROBLEMS, OR FOR NON-LINEAR ONES IN WHICH "UPSTREAM" COEFFICIENTS ARE ACCURATE ENOUGH.
  - THE LATTER ERROR IN NON-LINEAR PROBLEMS CAN BE REMOVED BY ITERATION ON THE LINE BEFORE PASSING DOWNSTREAM.

HTE 1 23	9/15	NEARLY PARABOLIC PROBLEMS

- NATURE: ALL a_E'S, SAY, ARE MUCH SMALLER THAN a_W'S.
- OCCURRENCE: AS FOR PARABOLIC PROBLEMS, BUT AT LOWER REYNOLDS (OR PECLET) NUMBERS.
- SUITABLE LINE-BY-LINE PROCEDURE:
  - AS FOR PARABOLIC, EXCEPT THAT MORE THAN ONE SWEEP WILL BE NEEDED.
  - "LOOPING INTEGRATION" WILL PROBABLY BE MORE ECONOMICAL THAN SWEEPS WHICH COVER THE DOMAIN FROM END TO END.

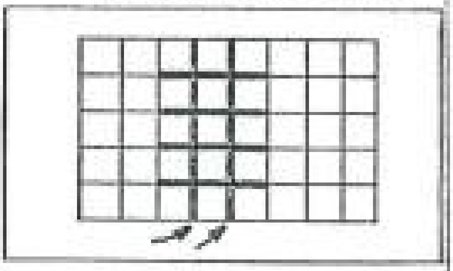


- CONSEQUENCES: CONVERGENCE MORE RAPID THAN FOR GENERAL ELLIPTIC PROBLEM.

HTE 1 23	10/15	THE TWO-VARIABLE ALGORITHM, TVA.

- NATURE: SOLVES $x_i = Ax_{i+1} + Bx_{i-1} + C + Dy_i$

  $y_i = ay_{i+1} + by_{i-1} + c + dx_i$ SIMULTANEOUSLY.
- APPLICABILITY: SOLVES FOR 2 LINES AT THE SAME TIME.



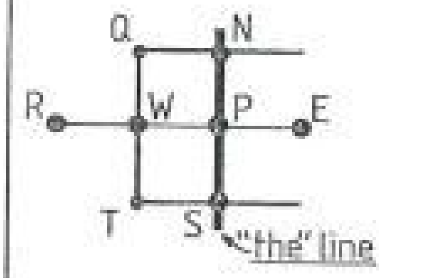
- DETAILS: • REDUCE FIRST TO FORM:

  $x_i = A'x_{i+1} + C' + D'y_{i+1}$

  $y_i = a'y_{i+1} + c' + d'x_{i+1}$, WHERE A', C', ETC. ARE DEDUCIBLE FROM A, B, C,... • THEN SOLVE FOR x_i, y_i BY RECURRENCE, STARTING FROM THE LARGE-INDEX END.
- ADVANTAGES: • MORE RAPID CONVERGENCE AT COST OF GREATER STORAGE. • ESPECIALLY USEFUL FOR NEAR-PARABOLIC PROBLEMS.

HTE 1 23	11/15	THE PARTIAL-ELIMINATION ALGORITHM, PEA.

- NATURE: IN THE EQUATION FOR ϕ_P, ϕ_W IS REPLACED BY $\Sigma a_W \phi_{nW} + b_W$; AND ϕ_E LIKEWISE.
- APPLICABILITY: USEFUL WHEN E-W LINKS ARE STRONG AND TRAVERSE IS IN NS DIRECTION.



- DETAILS: $\phi_P = a'_{PN}\phi_N + a'_{PS}\phi_S + b'_P$

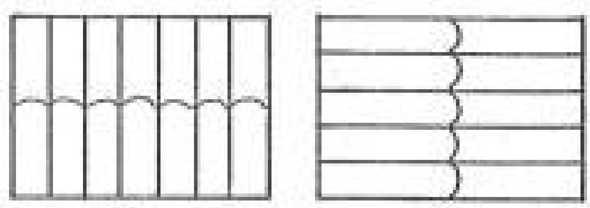
  $a'_{PN} \equiv a_{PN}/[1 - a_{PW}a_{WP} - a_{PE}a_{EP}]$

  $a'_{PS} = a_{PS}/[1 - a_{PW}a_{WP} - a_{PE}a_{EP}]$

  $b'_P = f\{b_P, a_{PW}, \ldots\ \phi_R, \phi_Q, \phi_T, \ldots\}$
- ADVANTAGES: FASTER CONVERGENCE.
- EXTENSION: THE LATERAL INFLUENCES MAY EXTEND TO R AND BEYOND.

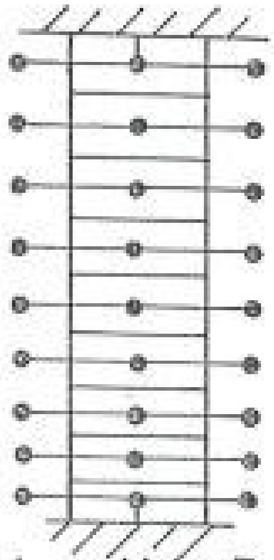
HTE 1 23	12/15	STRIP-WISE BLOCK RELAXATION

- NATURE: ϕ'S ARE SOUGHT, DEPENDENT ON (SAY) x ALONE, WHICH WILL REDUCE TO ZERO THE TOTAL ERROR OF THE CELLS IN EACH (SAY) y-DIRECTION STRIP; OBTAINED BY TDMA.
- APPLICABILITY: EXTENSIVE FIXED-FLUX BOUNDARIES LEAVE LITTLE POSSIBILITY OF ERROR ELIMINATION; BLOCK RELAXATION DOES AT LEAST CONNECT WITH ALL POINTS ON BOUNDARIES, AND THEREFORE TOUCHES THOSE WITH FIXED-VALUE INFLUENCES.
- ALTERNATION OF DIRECTION: IT IS OFTEN DESIRABLE TO MAKE ADJUSTMENTS FOR x-WISE AND y-WISE STRIPS IN TURN.



HTE 1 23	13/15	BLOCK RELAXATION; DETAILS.

- THE FDE FOR EACH CELL IN A STRIP IS WRITTEN AS:
  (FLUX FROM LEFT)-(FLUX TO RIGHT)
  +(FLUX FROM BELOW)-(FLUX TO TOP)
  + INTERNAL SOURCE = 0.
- ADDITION FOR WHOLE STRIP LEADS TO: Σ ALL FLUXES FROM LEFT
  -Σ ALL FLUXES TO RIGHT
  + FLUX FROM VERY BOTTOM
  - FLUX TO VERY TOP
  +Σ ALL INTERNAL SOURCES
  = 0.
- IN-STRIP FLUXES HAVE CANCELLED.



HTE 1 23	14/15	BLOCK RELAXATION; FURTHER DETAILS.

- EACH NET FLUX FOR A STRIP IS WRITTEN AS $F_* + a\phi'_L + b\phi'_M + c\phi'_R$, WHERE a, b, c EXPRESS EFFECTS OF DIFFUSION AND CONVECTION AS IN LECTURE 14, AND ϕ'_L, ϕ'_M AND ϕ'_R ARE THE SOUGHT-FOR INCREMENTS.
- EACH SOURCE IS WRITTEN AS $S_* + d\,\phi'_M$.
- THE RESULT IS AN EQUATION OF THE FORM:
  $\phi'_M = A\,\phi'_L + B\,\phi'_R + C$ FOR EACH STRIP.
- THE WHOLE SET OF EQUATIONS CAN THEN BE SOLVED BY A TDMA TRAVERSE NORMAL TO THE STRIP LENGTH; THEN THE ϕ'S ARE ADDED TO THE ϕ_*'S.
- NOTE THAT THE F_*'S MUST BE CALCULATED ONLY FOR STRIP BOUNDARIES, NOT FOR INTERNAL ONES.

HTE 1 23	15 15	FINAL REMARKS

- IT IS RARELY USEFUL TO USE POINT-BY-POINT PROCEDURES.
- THE LINE-BY-LINE PROCEDURES ARE BETTER; BUT THEY NEED AUGMENTATION BY THE FOLLOWING FURTHER DEVICES:
  - BLOCK RELAXATION, ESPECIALLY FOR FIXED-FLUX BOUNDARIES;
  - TVA, ESPECIALLY FOR STRONGLY-LINKED LINES.
  - PEA, IN VARIOUS FORMS.
  - WHOLE-FIELD PROCEDURES, SUCH AS THAT OF PANEL 20.7.
- PARABOLIC AND NEAR-PARABOLIC PROBLEMS, WHEN THE SOLUTION ORDER IS APPROPRIATE, PERMIT ESPECIALLY SIMPLE SOLUTION.

HTE 1 24	1/15	LECTURE 24. TWO-DIMENSIONAL HYDRODYNAMIC PROBLEMS.

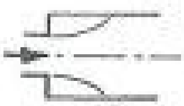
- CONTENTS:
  - THE PROBLEM OF COUPLING.
  - SIVA, POINT BY POINT.
  - SIMPLE, LINE BY LINE.
  - SPECIAL FEATURES:
    - WALL EFFECTS.
    - DISTRIBUTED RESISTANCES.
    - OBSTACLES.
- NOTE:
  - COMPRESSIBILITY EFFECTS ARE NOT DISCUSSED.
  - STEADY AND UNSTEADY PROBLEMS ARE HANDLED WITHOUT DISTINCTION.

HTE 1 24	2/15	THE PROBLEM IN PHYSICAL & MATHEMATICAL TERMS.

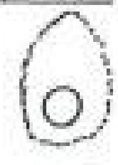
EXAMPLES OF PROCESSES TO BE PREDICTED:

- IMPINGEMENT OF JET ON WALL.

- SUDDEN ENLARGEMENT.



- PROCESS OF DROPLET BURNING

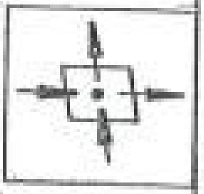


- FLOW AROUND OR BETWEEN AEROFOILS.

MATHEMATICAL FEATURES:

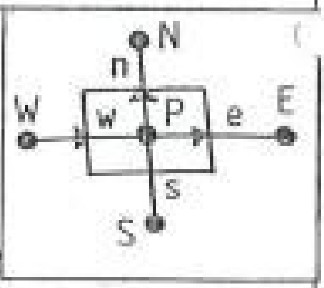
- 2 (OR 3) MOMENTUM EQUATIONS ARE COUPLED VIA CONTINUITY.
- EQUATIONS ARE NECESSARILY NON-LINEAR; SO ITERATION WILL BE NEEDED.
- INTERACTIONS EXIST BETWEEN ENERGY AND CONCENTRATION EQUATIONS AND THOSE FOR u, v, w, E.G. VIA μ, ρ.

HTE 1	3	POINT-BY-POINT PROCEDURES;
24	15	GENERAL FEATURES OF SIVA.



- REFERENCES TO EARLIER LECTURES: 19.13, 14.14.
- REMINDER OF MAIN FEATURES OF SIVA FOR 2D FLOW:
  - SIVA FOCUSSES ATTENTION ON PRESSURE.
  - p IS ADJUSTED, POINT BY POINT, SO AS TO SATISFY CONTINUITY FOR THE CELL.
  - IN THE MOMENTUM EQUATIONS USED FOR DEDUCING INFLUENCE OF p ON MASS FLOW RATE, ONLY THE VELOCITIES AT THE POINTS INDICATED ARE REGARDED AS VARIABLES.
- PROCEDURE OF ADJUSTMENT: • p_P IS ADJUSTED;
  - THEN u'S AND v'S FOR NEIGHBOUR POINTS ARE ADJUSTED TO CORRESPOND.

HTE 1	4	DETAILS OF SIVA, FOR 2D FLOW.
24	15	



- EQUATIONS:

$$\left.\begin{aligned} u_w &= L(u_e, p_P), \\ u_e &= L(u_w, p_P), \\ v_n &= L(v_s, p_P), \\ v_s &= L(v_n, p_P), \end{aligned}\right\} \leftarrow \text{MOMENTUM}$$

AND $L(u_w, u_e, v_n, v_s, p_P) = 0$. ← CONTINUITY

- N.B. p_P IN LAST EQUATION ENTERS VIA ρ ONLY FOR COMPRESSIBLE FLOW AND PRIMARILY THROUGH TRANSIENT TERMS.
- ALGEBRAIC MANIPULATION LEADS TO:

  p_P = KNOWN FUNCTION; $u_w = L(p_P)$; $u_e = L(p_P)$, $v_n = L(p_P)$, $v_s = L(p_P)$.
- VISITING PROCEDURE • UNDER- OR OVER-RELAXATION.
- ITERATION TO CONVERGENCE BY ERROR-SPREAD TO BOUNDARIES.

HTE 1 24	5/15	DISCUSSION OF SIVA

- HOW SIVA WORKS: • IF u's AND v's ARE SUCH AS TO BRING TOO MUCH FLUID INTO THE CONTROL VOLUME (I.E. IF A MASS SINK WOULD BE NEEDED TO ABSORB IT), p_P WILL RISE.
  - THE p_P INCREASE WILL INCREASE u_e AND v_n, AND DECREASE v_s AND u_w; THESE EFFECTS ARE IN THE RIGHT DIRECTION.
  - AT THE END OF AN ADJUSTMENT OF p_P, u_w, u_e, v_n, u_s, CONTINUITY IS SATISFIED FOR CELL P; BUT ERRORS HAVE BEEN CREATED AT NEIGHBOUR CELLS.
- THE PROPERTIES OF SIVA: AS FOR POINT-BY-POINT HEAT-CONDUCTION PROCEDURES; COEFFICIENTS DEPEND ON u's AND v's, INTRODUCING NON-LINEARITY.

HTE 1 24	6/15	LINE-BY-LINE SOLUTION PROCEDURES; GENERAL.

- REFERENCES TO EARLIER LECTURES: 20.2 ET SEQ.
- MAIN OPTIONS:
  - LINE-BY-LINE VERSION OF SIVA, INVOLVING USE OF A SPECIAL 4-VARIABLE ALGORITHM (WHICH IS NOT HARD TO DEVISE).
  - "SIMPLE," IN WHICH u's, v's AND p's ARE SOLVED SEPARATELY.
  - INTERMEDIATE PROCEDURES.
- PURPOSE OF FOLLOWING DISCUSSION: TO ILLUSTRATE IN A 2D CONTEXT THE DISCUSSION OF LECTURE 20, PANELS 12 TO 15.

HTE 1 24	7/15	DETAILED DESCRIPTION OF SIMPLE, (SEE PANELS 20.13, 14, 15 FOR EARLIER ACCOUNT)

- CALCULATION OF COEFFICIENTS: • A 2D STORE IS PROVIDED FOR EACH OF THE FIVE COEFFICIENTS OF EACH POINT.
  - ONLY ONE VARIABLE IS DEALT WITH AT A TIME; SO ONLY ONE SET OF STORES IS PROVIDED.
- SOURCE TERMS: • TWO NUMBERS ARE STORED FOR EACH POINT, PERMITTING THE SOURCE TO BE EXPRESSED AS A LINEAR FUNCTION: $s = a + b\,\phi_P$.
- COEFFICIENTS AND SOURCE CONSTANTS ARE HELD FIXED WHILE, FOR THE VARIABLE IN QUESTION, ADI SWEEPS ARE USED TO PROVIDE MORE EXACT VALUES OF THE ϕ's AT GRID NODES.

HTE 1 24	8/15	DESCRIPTION OF SIMPLE, CONTINUED. THE u_*, v_* SEQUENCES.

- THE PRESSURE IS GUESSED: USUALLY THE VALUES ALREADY IN STORE ARE TAKEN. THIS IS THE p_* FIELD.
- THE u_* SEQUENCE: • THESE p's PERMIT CALCULATION OF THE u-SOURCES.
  - THE FDE's FOR u CAN BE SET UP FOR THE WHOLE FIELD.
  - THEN ADI SWEEPS (USUALLY A SINGLE PAIR) CAN BE MADE TO SOLVE THE FDE's).
  - THE RESULTS ARE TERMED u_*'s.
- THE v_* SEQUENCE: • THE SAME p's PERMIT SOURCES OF v-MOMENTUM TO BE ESTABLISHED.
  - SOLUTION OF THE FDE's LEADS TO THE v_* FIELD.

HTE 1 24	9 15	DESCRIPTION OF SIMPLE, CONTINUED. CORRECTION OF VELOCITIES.

- THE u_* AND v_* FIELDS ARE NOT COMPATIBLE WITH CONTINUITY: THE INCOMPATIBILITY CAN BE EXPRESSED, FOR EACH CELL, AS A CONTINUITY ERROR.
- THE "PRESSURE-CORRECTION EQUATION" IS SET UP, WITH COEFFICIENTS FROM THE MOMENTUM EQUATIONS, AND "SOURCES" FROM THE CONTINUITY ERRORS (SEE 16.8).
- A SOLUTION FOR p' IS OBTAINED BY ADI OVER THE WHOLE 2D FIELD, USUALLY BY MULTIPLE ITERATIONS.
- CORRESPONDING VELOCITY ADJUSTMENTS ARE THEN MADE.

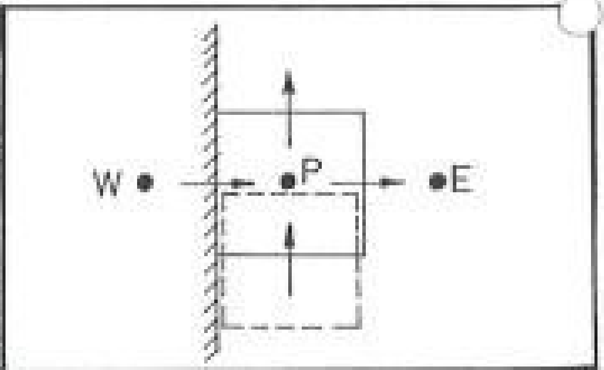
HTE 1 24	10 15	DESCRIPTION OF SIMPLE; UNDER-RELAXATION.

- THE FULL p' CORRECTION MAY BE ADDED TO p_*; MORE USUALLY ONLY A FRACTION < 1 IS ADDED. THIS IS CALLED UNDER-RELAXATION OF PRESSURE.
- THE FULL u', v' CORRECTIONS SHOULD HOWEVER BE ADDED TO u_* AND v_*; OTHERWISE, CONTINUITY WOULD NOT BE SATISFIED.
- WHEN u_* AND v_* ARE OBTAINED FROM p_*, OFTEN HEAVY UNDER-RELAXATION IS EMPLOYED; THIS IS NEEDED FOR CONVERGENCE.
- "HEAVY" MEANS α (OF PANEL 20.5) AS LOW AS 0.1.

HTE 1	11	DESCRIPTION OF SIMPLE;
24	15	ITERATIONS.

- BETWEEN EACH u_*, v_*, p', u', v', p_* CYCLE, SOLUTIONS OF OTHER EQUATIONS MAY BE INTERSPERSED. (N.B. THEY SHOULD BE, IF THERE ARE INTERACTIONS BETWEEN THEM AND THE HYDRODYNAMICS; OTHERWISE, THESE EQUATIONS SHOULD BE SOLVED ONLY AFTER SOLUTIONS FOR u, v, p HAVE BEEN OBTAINED).
- ITERATIONS MAY CEASE WHEN ALL ERRORS IN CONTINUITY, MOMENTUM, ENERGY, ETC., FOR ALL CELLS IN THE FIELD, ARE SMALLER THAN PRE-ASSIGNED VALUES.
- CONVERGENCE IS USUALLY OBTAINABLE FOR LOW UNDER-RELAXATION FACTORS OF u AND v, E.G. LESS THAN 0.5

HTE 1	12	SOME SPECIAL FEATURES:
24	15	TREATMENT OF WALLS.



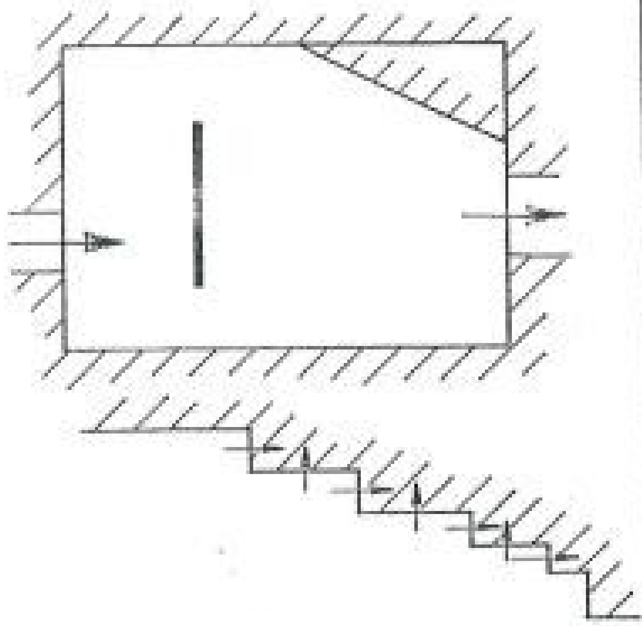
- HEAT CONDUCTION (AS EXAMPLE):
  - CONDUCTION FROM P TO E CAN BE HANDLED IN TERMS OF "THE" CONDUCTIVITY BETWEEN THE NODES.
  - FOR THE $P \sim W$ REGION, OFTEN CONDUCTIVITY VARIES RAPIDLY; A SPECIALLY-CALCULATED VALUE MAY BE USED. $(\equiv (x_P - x_W)/\int_W^P \lambda^{-1} dx)$.
  - ALTERNATIVELY, THE COEFFICIENT c_W MAY BE PUT TO ZERO, AND THE WALL EFFECT CALCULATED VIA SOURCE TERMS.
- FOR VELOCITY, AND OTHER VARIABLES, SIMILAR DEVICES ARE EMPLOYED.

HTE 1 24	13 — 15	SOME SPECIAL FEATURES; DISTRIBUTED RESISTANCES.

- PHYSICAL PROCESSES TO BE SIMULATED:
  - LINEAR RESISTANCE, FIXED OBSTACLE:
    $S_u = -F_x u,\ S_v = -F_y v.$
  - LINEAR RESISTANCE, MOVING OBSTACLE:
    $S_u = -F_x(u - u_{res}),\ S_v = -F_y(v - v_{res}).$
  - GENERAL LINEARISED RESISTANCE:
    $S_u = a_u + b_u u,\ S_v = a_v + b_v v.$
- INCORPORATION IN FDE'S;
  THROUGH a AND b OF PANEL 14.13.
- APPLICATION: THIS DEVICE PERMITS SIMPLE TO BE USED FOR POROUS MEDIA, HEAT EXCHANGERS, ETC.

HTE 1 24	14 — 15	SOME SPECIAL FEATURES; OBSTACLES AND SLOPING WALLS.

- GENERAL IDEA:
  AN OBSTACLE IS A RESISTANCE, REDUCING THE NORMAL VELOCITY TO ZERO.
- HOW TO REDUCE u TO ZERO:
  PUT $a_u = 0.,\ b_u = -10^{+20}.$
- REPRESENTATION OF SLOPING WALLS: PUT u AND v TO ZERO ALONG "STAIRCASE".



HTE 1 24	15 15	FINAL REMARKS

- THE COUPLING OF THE EQUATIONS IS BEST HANDLED BY SIVA.
- HOWEVER, THIS IS EASY ONLY IN POINT-BY-POINT FORM; THE LINE-BY-LINE VERSION HAS NOT YET BEEN DEVELOPED.
- SIMPLE HANDLES THE COUPLING PROBLEM BY A GUESS-AND-CORRECT PROCEDURE; PROVIDED u AND v ARE UNDER-RELAXED, SO THAT INCORRECT PRESSURES DO NOT LEAD TO LARGE CONTINUITY ERRORS (AND SO STILL LARGER PRESSURE ERRORS) CONVERGENCE IS ATTAINED.
- THE LINEARISED-SOURCE FACILITY PROVIDES AN EASY MEANS FOR SIMULATING RESISTANCES, OBSTACLES AND BOUNDARIES.

HTE 1 25	1/15	LECTURE 25. IMPROVED PROCEDURES FOR HYDRODYNAMIC PROBLEMS.

- CONTENTS:
  - DEFECTS OF SIMPLE.
  - SNIP.
  - SIMPLER.
  - SIMPLEST.
  - OTHER IMPROVEMENTS.
- NOTE: THE FRONTIER OF THE SUBJECT HAS NOW BEEN REACHED.

HTE 1 25	2/15	DEFECTS OF SIMPLE; USE OF STORAGE.

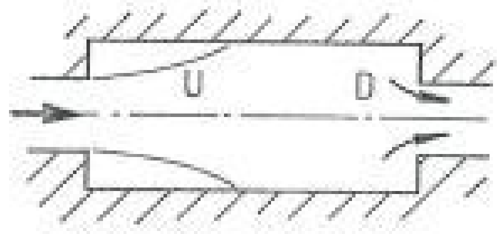
- COEFFICIENTS:

  A 2D ARRAY IS USED FOR EACH OF FOUR COEFFICIENTS, FOR THE WHOLE FIELD; YET ONLY THOSE APPROPRIATE TO A SINGLE LINE ARE USED AT ANY ONE TIME IN THE ADI.
- AUXILIARY PROPERTIES:

  2D ARRAYS ARE ALSO USED FOR ρ, μ, ETC., SO AS TO MAKE IT EASY TO CALCULATE THESE COEFFICIENTS.
- CRITICISM: THE "WHOLE-FIELD" APPROACH IS VERY EXPENSIVE.

HTE 1 25	3 — 15	DEFECTS OF SIMPLE; WASTED COMPUTATIONS.

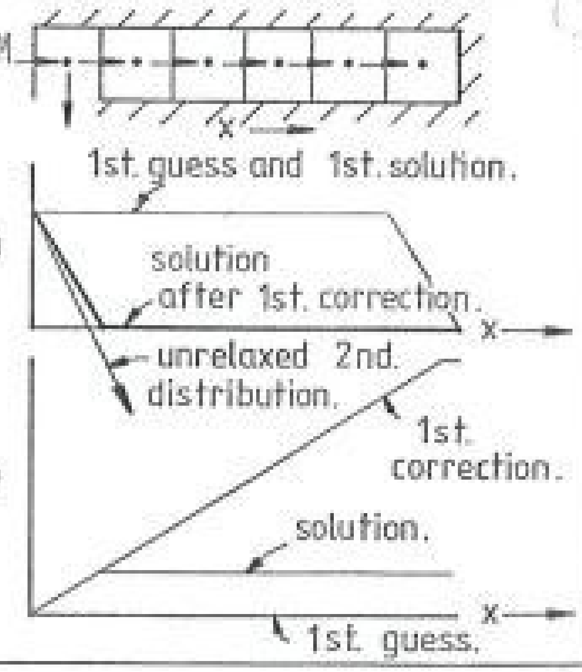
- MANY PRACTICAL FLOWS HAVE A GENERAL UPSTREAM-DOWNSTREAM CHARACTER, E.G. COMBUSTION CHAMBERS, TURBINES, SHIPS, AIRCRAFT, ETC.
- INFLUENCES THEREFORE PASS MAINLY DOWNSTREAM.



- COMPUTATIONS MADE IN THE DOWNSTREAM REGION (D) BEFORE THOSE OF THE UPSTREAM REGION (U) HAVE SETTLED DOWN, ARE LARGELY WASTED.
- SOLUTIONS FOR T IN A UNIFORM-PROPERTY SITUATION ARE WASTEFUL IF STARTED BEFORE THE COMPLETE FLOW FIELD HAS BEEN COMPUTED.

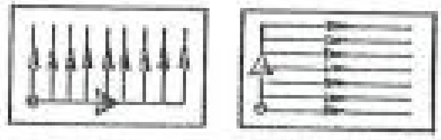
HTE 1 25	4 — 15	DEFECTS OF SIMPLE; IMPROPER PRESSURE FIELDS.

- CONSIDER 1D STEADY-FLOW PROBLEM INDICATED: →
- THE p' EQUATION GIVES A SLOPING p' ∿ x VARIATION CORRESPONDING TO MASS SOURCE AT RH END.
- IF NOT RELAXED, THIS LEADS TO VERY LARGE OPPOSITELY-DIRECTED VELOCITIES.
- UNDER-RELAXATION, AND MANY ITERATIONS, ARE NEEDED.



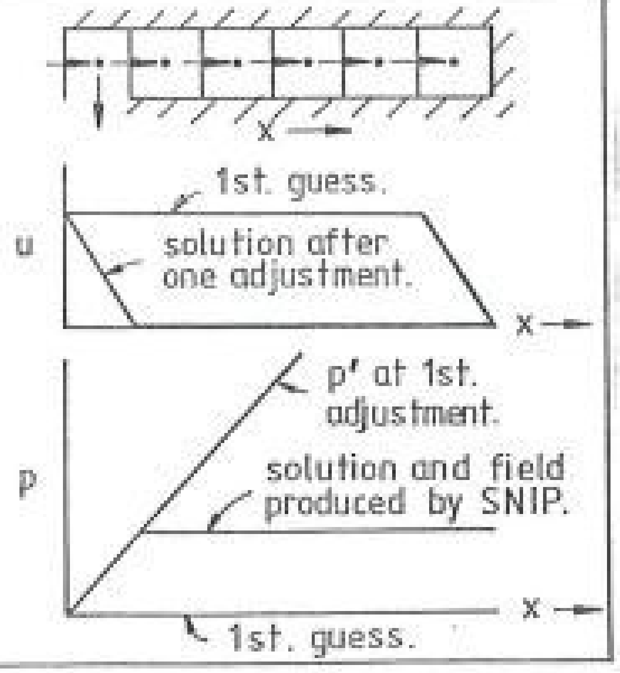
HTE 1 25	5/15	ALTERNATIVE SOURCES OF p_*: SNIP (≡ START WITH NEW INTEGRATION FOR PRESSURE), SPALDING, 1975.

- SIMPLE EMPLOYS FOR THE NEXT STEP THE PRESSURE FIELD $p_* + \alpha p'$, WHERE α IS A RELAXATION FACTOR AND p' IS THE RESULT OF SOLVING THE POISSON VELOCITY-CORRECTION EQUATION.
- THE QUESTION: WHAT IS THE BEST ESTIMATE WHICH CAN BE MADE OF THE PRESSURE FIELD?
- SNIP ANSWERS: • THAT WHICH MOST NEARLY SATISFIES THE TWO MOMENTUM EQUATIONS, DEDUCIBLE BY STEP-BY-STEP INTEGRATION.
  - TWO DIFFERENT p FIELDS CAN BE DEDUCED ACCORDING TO THE EQUATION CHOSEN (SEE RIGHT →).
- EITHER, OR THE MEAN, MAY BE CHOSEN AS p_*.

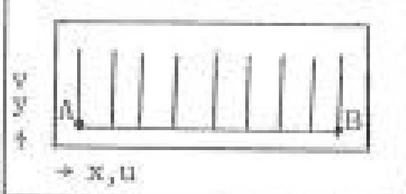


HTE 1 25	6/15	SNIP APPLIED TO THE 1D PROBLEM OF PANEL 4.

- LET u_*, CONTINUITY ERRORS, POISSON EQUATION, u', u CALCULATIONS BE AS IN SIMPLE.
- SNIP THEN DISREGARDS THE RESULTING p', AND THE EARLIER p_*, FIELDS.
- SNIP DEDUCES p FROM THE ADJUSTED u FIELD (WHICH SATISFIES CONTINUITY).
- THIS IS THE SOLUTION FOR 1D.
- NO FURTHER ADJUSTMENT IS NEEDED.



HTE 1 25	7/15	SNIP APPLIED TO 2D PROBLEMS; DETAILS.



- LINES ALONG WHICH SNIP IS TO BE APPLIED MUST BE CHOSEN ARBITRARILY, E.G. AS IN SKETCH.
- $p - p_A$ ALONG AB IS OBTAINED FROM THE FDE's FOR u, WRITTEN AS: $p_e = p_w + L(u_p, u_N, u_S, u_E, u_W)$.
- $p - p_A$ FOR ALL OTHER POINTS IN THE FIELD ARE THEN OBTAINED FROM THE FDE's FOR v, WRITTEN AS: $p_n = p_s + L(v_p, v_N, v_S, v_E, v_W)$.
- THE RESULTING p FIELD IS INSERTED IN THE REMAINING (I.E. NON-AB) FDE's FOR u, WHICH ARE THEN SOLVED BY ADI.
- THE RESULTING u's, TOGETHER WITH THE UNCHANGED u_{AB}'s AND THE UNCHANGED v's CONSTITUTE THE NEW u_*, v_* FIELDS.
- THESE IMPLY CONTINUITY ERRORS, WHICH WILL BE ELIMINATED BY THE NEXT p' SOLUTION.

HTE 1 25	8/15	SNIP APPLIED TO 2D PROBLEMS; DISCUSSION

- MUCH COMPUTATION (ADI OF v) HAS BEEN SAVED.
- NO PAST (ERRONEOUS) PRESSURE FIELDS PLAY ANY PART.
- JUDICIOUS CHOICE OF RIB AND SPINE LOCATION MAY ACCELERATE CONVERGENCE.
- THE METHOD MAY BE APPLIED TO 3D FLOWS ALSO.

HOWEVER:

- THE ARBITRARY RIB-SPINE CHOICE IS UNAESTHETIC.
- IF OBSTACLES INTERSECT AN INTEGRATION PATH, THE METHOD FAILS.
- VERY LARGE RIB-TO-RIB PRESSURE DIFFERENCES MAY BUILD UP WHEN THE INITIAL u, v FIELDS ARE ERRONEOUS.
- IN COMPRESSIBLE FLOWS, IT IS NOT CLEAR AT WHAT POINT DENSITIES SHOULD BE UPDATED.

HTE 1 25	9/15	ALTERNATIVE SOURCES OF p_*: SIMPLER (≡ SIMPLE REVISED), PATANKAR (1979).

- SIMPLER, LIKE SNIP, GETS THE p_* FIELD FROM THE CONTINUITY-SATISFYING u, v FIELDS.
- SIMPLER OBTAINS p_* BY SOLVING AN ADDITIONAL POISSON EQUATION BY ADI.
- THE 2D EQUATION RESULTS FROM INSERTING "PSEUDO-VELOCITIES" $\hat{u}$, $\hat{v}$ IN THE CONTINUITY EQUATION AND HENCE GENERATING CORRESPONDING "PSEUDO-ERRORS" IN CONTINUITY; OTHERWISE THE p_* EQUATION IS IDENTICAL WITH THE p' EQUATION.
- SIMPLER THUS DOES MORE COMPUTATIONAL WORK THAN SIMPLE, WHERE-AS SNIP DOES LESS; BUT IT IS FREE FROM ARBITRARINESS OF APPLICATION PATTERN; AND IT CAN HANDLE THE PRESENCE OF OBSTACLES.
- SIMPLER AND SNIP PRODUCE THE SAME RESULTS FOR 1D INCOMPRESSIBLE FLOWS.

HTE 1 25	10/15	FURTHER IMPROVEMENTS TO SIMPLE: SIMPLEST (SIMPLE-SHORTENED), SPALDING, 1979

- MAIN FEATURES: (1) CEASE TO USE ADI FOR u_*, v_* COMPUTATIONS; REPLACE BY JACOBI, POINT-BY-POINT. (2) USE POISSON EQUATION FOR p' TO GET u' AND v', TO BE ADDED TO u_* AND v_*, AND GET p FROM (p_*+p'). (3) USE SNIP WITH JUDICIOUS CHOICE OF RIB-SPINE LOCATION TO PROCURE PRESSURE DIFFERENCES FOR MOMENTUM SOURCES.
- PRESENT STATUS:
  1) JACOBI PBP PROVIDES FASTER CONVERGENCE WITH LITTLE NEED FOR UNDER-RELAXATION.
  2) THIS FEATURE IS CONVENTIONAL.
  3) PRELIMINARY RESULTS ARE SATISFACTORY.

HTE 1 25	$\frac{11}{15}$	DISCUSSION OF SIMPLEST; FEATURE (1).

- THE CONVECTION TERMS IN THE u_* AND v_* EQUATIONS ACT AS ERROR-AUGMENTORS, WHEN SIMULTANEOUS SOLUTION IS USED.
- THE p'-COEFFICIENTS HAVE BEEN FORMED BY NEGLECTING MOST OR ALL CONVECTIVE INTERACTIONS: SO THE p' FIELDS OVER-CORRECT THE MOMENTUM IMBALANCES. HENCE THE NEED FOR UNDER-RELAXATION.
- THE $\hat{u}, \hat{v} \rightarrow p_*$ STAGE OF SIMPLER CAN BE VIEWED AS EQUIVALENT TO A JACOBI PBP u_*, $v_* \rightarrow p'$ STAGE.
- PROBABLY IT WILL BE USEFUL, IN FUTURE, TO DEAL WITH DIFFUSIVE TRANSPORT SIMULTANEOUSLY AND CONVECTIVE TRANSPORT POINT-BY-POINT.

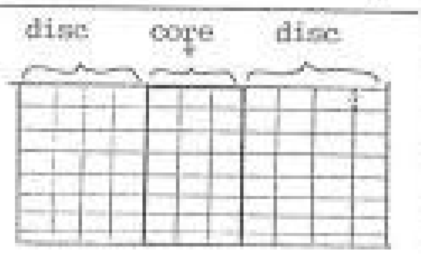
HTE 1 25	$\frac{12}{15}$	DISCUSSION OF SIMPLEST; FEATURES (2) AND (3).

(2)
- THE p'-SOLUTION STAGE IS THE MAIN MEANS OF ACCOUNTING FOR ELLIPTIC EFFECTS; IT IS WORTH TAKING ESPECIAL CARE OVER ITS EFFICIENCY.
- ρ IS THE DENSITY WHICH IS USED FOR ALL CONVECTION-TERM AND CONTINUITY-ERROR COMPUTATIONS; IT IS RESPONSIVE TO THE ABSOLUTE VALUE OF p, NOT JUST TO p-DIFFERENCES (LIKE u AND v).

(3)
- SINCE IT TAKES NO MORE TIME TO SOLVE PBP FOR u_P THAN TO GET $p_e - p_w$ FROM THE SAME EQUATION, THE EXTENT TO WHICH SNIP IS USED, AND THE NUMBER OF PBP u AND v ADJUSTMENTS IS CONSEQUENTLY DIMINISHED, CAN BE SETTLED BY COMPUTER-CODING CONSIDERATIONS.

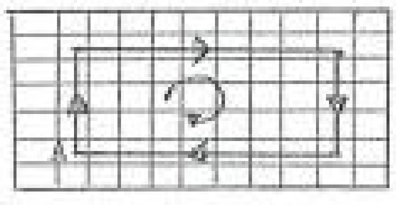
HTE 1 25	13/15	OTHER IMPROVEMENTS: USE OF STORAGE

- THE NEED: REALISM REQUIRES FINE GRIDS; COMPUTER CORES ARE TOO SMALL, ESPECIALLY FOR 3D PROBLEMS.
- A SOLUTION: USE SECONDARY STORAGE (DISC, TAPE) AND RETURN IN CORE ONLY WHAT IS NEEDED DURING THE CURRENT COMPUTATION SEQUENCE.
- PARTICULAR EXAMPLE:



- ALLOW IN-CORE STORAGE FOR ONLY THREE STRIPS (OR SLABS) OF INTEGRATION DOMAIN.
- MOVE EQUATION-SOLVER REPEATEDLY THROUGH INTEGRATION DOMAIN.
- IF POSSIBLE, RETAIN p' SOLVER IN CORE.

HTE 1 25	14/15	OTHER IMPROVEMENTS: CIRCULATION ADJUSTMENTS.



- NATURE: • PERIODICALLY INTEGRATE FOR PRESSURE AROUND SOME (ARBITRARY, BUT SIGNIFICANT) CIRCUIT IN THE FIELD.
- THEN MAKE A CONTINUITY-SATISFYING ADJUSTMENT OF THE WHOLE FLOW FIELD (E.G. BY ADDING A STREAM-FUNCTION INCREMENT), WHICH WILL TEND TO REDUCE THE DISCREPANCY.
- EFFECTIVENESS: LARGE IMPROVEMENTS IN COMPUTATION SPEED (I.E. REDUCTION IN ERROR FOR GIVEN NUMBER OF ITERATIONS) CAN BE EFFECTED BY JUDICIOUS USE OF THIS DEVICE.

HTE 1 25	15 — 15	FINAL REMARKS

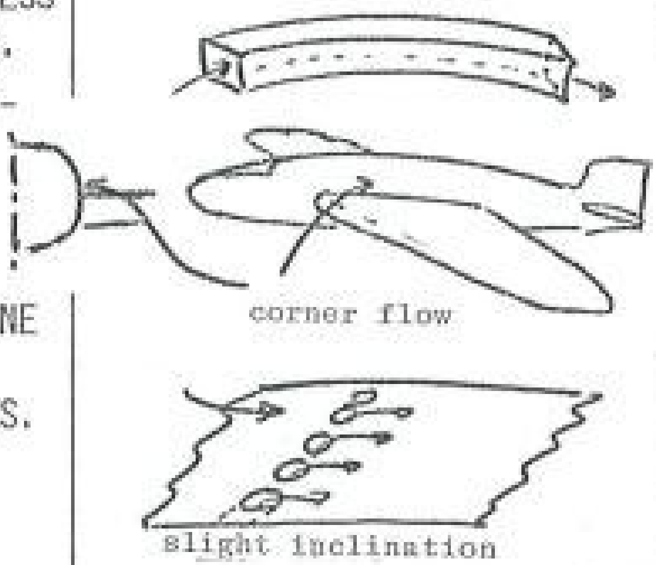
- SIMPLE HAS SERVED WELL; BUT IT IS DEFECTIVE IN VARIOUS RESPECTS, AND IS NOW BECOMING OBSOLETE.
- SNIP, SIMPLER, SIMPLEST AND NEAT ALL REPRESENT ATTEMPTS TO OVERCOME THE DEFICIENCIES OF SIMPLE, AND TO ARRIVE AT AN OPTIMAL METHOD.
- THE SEARCH IS FAR FROM ENDED; THERE IS MUCH SCOPE FOR FURTHER INNOVATION AND EXPLORATION.
- OFTEN THE "NEW IDEA" PROVES TO BE A REVERSION TO AN OLD ONE; THUS, CIRCULATION ADJUSTMENT WAS WHAT WAS DONE IN OLD STREAM-FUNCTION~VORTICY PROCEDURES.

HTE 1 26	1/15	LECTURE 26. 3D PARABOLIC & PARTIALLY-PARABOLIC PROBLEMS

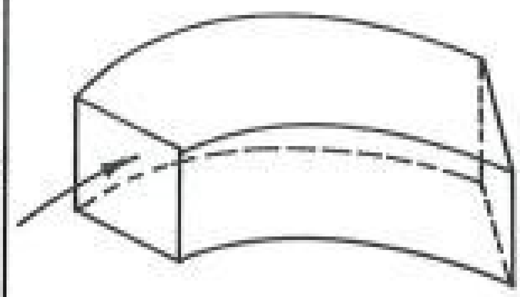
- CONTENTS:
  - FLOW PROCESSES IN QUESTION.
  - GENERAL FEATURES.
  - SOLUTION VIA SIVA.
  - SOLUTION VIA SIMPLE.
  - SPECIAL FEATURES OF PARTIALLY-PARABOLIC PROCESSES.
  - RECENT IMPROVEMENTS.
- NOTE: MUCH OF THE CONCEPTUAL AND DEVELOPMENT WORK ON THIS TOPIC HAS BEEN DONE IN THE HEAT TRANSFER SECTION.

HTE 1 26	2/15	EXAMPLES OF RELEVANT FLOW PROCESSES; PARABOLIC (REMINDER).

- ANY TIME-DEPENDENT PROCESS IN TWO SPACE DIMENSIONS.
- STEADY FLOW IN A MILDLY-CURVED DIFFUSER.
- FLOW NEAR THE JOIN OF WING AND FUSELAGE.
- FILM COOLING OF A TURBINE BLADE BY ROW OF INDIVIDUAL COOLANT HOLES.

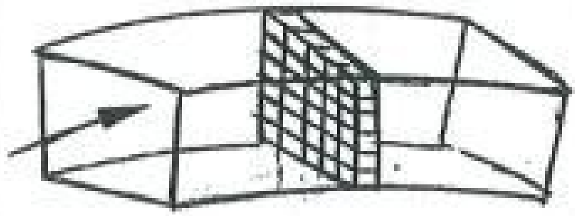


HTE 1 26	3 15	EXAMPLES OF RELEVANT FLOW PROCESSES; PARTIALLY-PARABOLIC (REMINDER).



- FLOW IN STRONGLY-CURVED DIFFUSER, OR COMPRESSOR PASSAGE.
- FLOW IN VICINITY OF SHIP'S STERN.
- 3D FILM COOLING WHEN THE INJECTION ANGLE IS LARGE.
- FLOW OVER AIRCRAFT FUSELAGE IN SUBSONIC FLIGHT.
- MOVEMENT OF STRATIFIED ATMOSPHERE OVER UNEVEN TERRAIN.

HTE 1 26	4 15	GENERAL FEATURES OF SOLUTION PROCEDURES; GRID AND STORAGE.



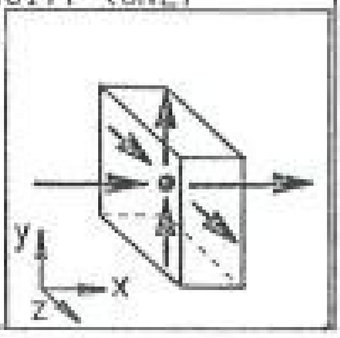
- ATTENTION IS CONCENTRATED ON A 2D ARRAY OF CELLS.
- THESE ARE MOVED DOWNSTREAM THROUGH THE FLOW IN "MARCHING-INTEGRATION" SWEEPS.
- AT ANY STAGE, SOLUTION TASK IS SIMILAR TO THAT OF LECTURE 24.
- INFLUENCES ARE ALL DOWNSTREAM EXCEPT FOR THAT OF PRESSURE (LECTURE 17).
- STORAGE IS 2D FOR ALL VARIABLES IN PARABOLIC FLOW, AND FOR ALL BUT PRESSURE, WHICH IS 3D, FOR PARTIALLY PARABOLIC.
- LATTER REQUIRES REPEATED SWEEPS; FORMER ONLY ONE.

HTE 1 26	5 15	GENERAL FEATURES OF SOLUTION PROCEDURES: UNCOUPLING OF PRESSURE FOR PARABOLIC FLOWS.

- THE PROBLEM: • IF PRESSURES ARE UNIFORM (OR OTHERWISE PRESCRIBED) OVER CROSS-SECTION, DISTRIBUTION OF FLOW BETWEEN CROSS-SECTION DIRECTIONS CANNOT BE SETTLED.
  - IF NON-UNIFORMITIES OF PRESSURE ARE ALLOWED TO INFLUENCE THE LONGITUDINAL MOMENTUM EQUATION, THE FLOW IS ELLIPTIC, AND MULTIPLE MARCHES ARE NEEDED.
- THE SOLUTION: • USE UNIFORM PRESSURE (OR PRESCRIBED NON-UNIFORMITY) IN LONGITUDINAL MOMENTUM EQUATIONS.
  - ALLOW NON-UNIFORMITIES ONLY IN CROSS-STREAM MOMENTUM EQUATIONS.
  - THIS PRACTICE "UNCOUPLES" LONGITUDINAL FROM CROSS-STREAM MOMENTUM EQUATIONS.

HTE 1 26	6 15	3D PARABOLIC PROBLEMS: NATURE OF SIVA PROCEDURE.

- ORIGIN: D B SPALDING, 1970; TESTED BY CARETTO, TATCHELL.
- PRECURSOR: VELOCITY ~ VORTICITY METHOD (CURR ET AL, 1969), IN WHICH PRESSURE UNCOUPLING WAS IMPLICIT AND SO NOT RECOGNISED.
- NATURE: • UNIFORM PRESSURE PRESUMED OVER CROSS-SECTION, CALCULATED SO AS TO ENSURE OVERALL CONTINUITY (ONLY FIXED-MASS-FLOW PROBLEMS WERE CONSIDERED)
  - THE 2D SIVA EQUATIONS ARE USED TO GIVE v AND w; THE u VELOCITIES ARE TAKEN AS FIXED.

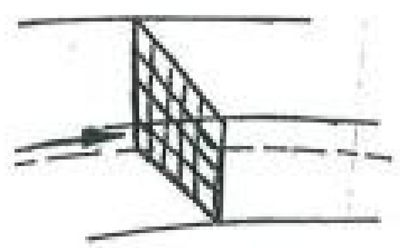


HTE 1 / 26	7 / 15	3D PARABOLIC PROBLEMS, SOLVED BY SIVA; DISCUSSION.

- THE METHOD WORKED SATISFACTORILY.
- THE POINT-BY-POINT NATURE OF THE PROCEDURE RENDERED IT TIME-CONSUMING FOR FINE GRIDS.
- FOR SUPERSONIC VELOCITIES IN THE LONGITUDINAL (x) DIRECTION, PRESSURE UNCOUPLING WAS NOT NEEDED, BECAUSE PRESSURE CHANGES TRANSMIT NO MASS-FLOW EFFECTS UPSTREAM IN ANY CASE.
- THE METHOD WAS NOT TRIED EXTENSIVELY, BEING ABANDONED (WISELY?) IN FAVOUR OF SIMPLE.

HTE 1 / 26	8 / 15	3D PARABOLIC PROBLEMS, SOLVED BY SIMPLE; NATURE.

- ORIGIN: S V PATANKAR, D B SPALDING, 1971; TESTED BY CARETTO, TATCHELL, SHARMA AND OTHERS.
- PRECURSOR: SIVA; THE LINE-BY-LINE PROCEDURE FOR 2D BOUNDARY LAYERS (S V PATANKAR, D B SPALDING).
- NATURE: • PRESSURE UNCOUPLING AS FOR SIVA AND (IMPLICITLY) VELOCITY ~ VORTICITY PROCEDURE.
  - SOLUTION OF x EQUATION USING UNIFORM PRESSURE ACROSS FLOW (DETERMINED FROM CONTINUITY IF FLOW CONFINED).
  - 2D STORAGE FOR COEFFICIENTS.
  - v_*, w_* FIELDS FROM GUESSED $p_*(y,z)$.
  - ADJUSTMENT OF v, w TO SATISFY CONTINUITY (POISSON EQUATION FOR p').



HTE 1 26	9/15	3D PARABOLIC PROBLEMS SOLVED BY SIMPLE; DISCUSSION.

- PUBLICATION: S V PATANKAR, D B SPALDING (1972).
- PROBLEMS SOLVED: • DUCT FLOWS, INCLUDING EFFECTS OF LATERAL BUOYANCY, CHEMICAL REACTION, MOVING WALL. • 3D FILM COOLING. • EXTERNAL BOUNDARY LAYERS, E.G. CORNER-FLOW PROBLEM. • 3D JETS.
- ADVANTAGES AS COMPARED WITH POINT-BY-POINT SIVA:
  • SHORTER COMPUTATION TIMES, ESPECIALLY FOR FINE GRIDS.
  • SIMPLER ALGEBRA. • SEPARATE TREATMENT OF u_*, v_*, w_*, $\bar{p}$, p' PROMOTES CONCEPTUAL CLARITY.
- COMPUTER PROGRAM: STABLER (≡STEADY THREE-DIMENSIONAL ANALYSIS OF BOUNDARY-LAYER EQUATIONS, REVISED).

HTE 1 26	10/15	3D PARABOLIC PROBLEMS; FURTHER REMARKS.

- IMPORTANCE OF ITERATION AT A FORWARD STEP; EARLY PROCEDURES TOOK SMALL FORWARD STEPS, AND PERFORMED ONLY ONE u_*, v_* + p' CYCLE PER STEP. THE RESULTS WERE SOMETIMES INACCURATE; AND COMPUTER TIMES WERE LONG.
- UTILITY OF JACOBI POINT-BY-POINT SOLUTION FOR u_* AND v_*: THIS HAS PROVED VERY ADVANTAGEOUS.
- PBP FOR w_*?: THIS HAS PROVED LESS ADVANTAGEOUS, PROBABLY BECAUSE VISCOUS (I.E. MOMENTUM-DIFFUSION) TERMS ARE MORE IMPORTANT THAN LATERAL (x- AND y-DIRECTION) TERMS.
- THREE-DIMENSIONAL BOUNDARY LAYERS ON AIRCRAFT, ETC: WHEN THE VISCOUS EFFECTS ARE CONFINED TO A VERY THIN LAYER, SOME STRESSES AND MOMENTUM TERMS BECOME UNIMPORTANT; NO POISSON EQUATION NEEDS TO BE SOLVED.

HTE 1 26	11/15	3D PARTIALLY-PARABOLIC PROBLEMS; PRACTICAL OCCURRENCE

- INTERNAL FLOWS:
  - CENTRIFUGAL PUMP AND COMPRESSOR ROTORS,
  - MIXING OF EXHAUST AND BY-PASS AIR IN JET ENGINES,
  - FLOW IN CURVED DUCTS AND DIFFUSERS,
  - FILM COOLING BY ARRAYS OF COOLANT HOLES.
- EXTERNAL FLOWS:
  - FLOWS AROUND SHIPS' HULLS;
  - FLOWS AROUND AIRCRAFT AND MISSILE FUSELAGES;
  - MOVEMENT OF ATMOSPHERE OVER UNDULATING TERRAIN.
- NOTE:
  ONLY STEADY FLOWS ARE IN QUESTION.

HTE 1 26	12/15	3D PARTIALLY PARABOLIC PROBLEMS; MAIN FEATURES OF PREDICTION PROCEDURE

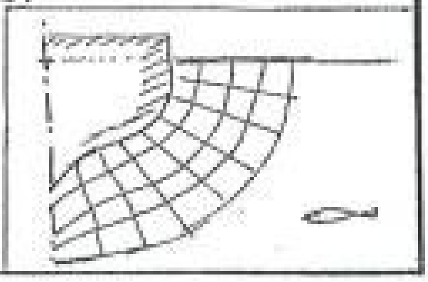
- ORIGINS: SPALDING (1971); PRATAP (1975); PRATAP AND SPALDING (1975, 1976).
- STORAGE: AS FOR 3D PARABOLIC, EXCEPT THAT:-
  - PRESSURE UNCOUPLING IS <u>NOT</u> EMPLOYED.
  - CHANGES TO u, v AND w ARE ALLOWED FOR IN PRESSURE-CORRECTION SEQUENCE.
  - THE u ADJUSTMENT CAUSES UPSTREAM CONTINUITY IMBALANCES; SO REPEATED MARCHING INTEGRATION IS NEEDED.
- COMPUTER TIME:
  - GREATER THAN FOR CORRESPONDING PARABOLIC PROBLEM, BECAUSE MARCHING INTEGRATION MUST BE REPEATED;
  - CAN BE REDUCED BY STORAGE OF SELECTED ADDITIONAL 3D VARIABLES, E.G. PRESSURE- CORRECTION COEFFICIENTS.

HTE 1 26	13/15	3D PARTIALLY-PARABOLIC PROBLEMS: SIMPLE PROCEDURE.

- NATURE: EACH INTEGRATION SWEEP IS EXACTLY AS FOR 3D PARABOLIC, EXCEPT FOR COUPLING OF PRESSURES.
- RESULT: CONVERGENCE IS OBTAINED; BUT IT MAY BE SLOW BECAUSE OF THE FACT THAT ERRORS AND ADJUSTMENTS CAN MOVE ONLY ONE PLANE UPSTREAM AT A TIME.
- IMPROVEMENTS:
    - USE LARGE x-STEPS AT FIRST, SMALL LATER.
    - ADJUST FARTHER UPSTREAM PRESSURES BY "ANTICIPATION FORMULAE" (APPROXIMATELY).
    - USE COARSER GRID FOR PRESSURE THAN FOR OTHER VARIABLES.

HTE 1 26	14/15	3D PARTIALLY-PARABOLIC PROBLEMS; RECENT DEVELOPMENTS.

- MAJOR USE HAS BEEN CONNECTED WITH FLOW AROUND SHIPS, WHERE DISTORTED-GRID FEATURES ARE NEEDED. (SEE SKETCH).
- IMPROVED CONVERGENCE CAN BE ACHIEVED BY USING A 3D (I.E. WHOLE-FIELD) SIMULTANEOUS-SOLUTION PROCEDURE FOR THE PRESSURE-CORRECTION EQUATION.
- OTHER USES ARISE IN TURBO-MACHINERY DESIGN.
- INCREASING USE OF SECONDARY STORAGE IN NEAT ARRANGEMENT (PANEL 25.13) ALLOWS u, v, w, k, ε VALUES TO BE STORED 3D, SO AS TO SAVE COMPUTATION.



HTE 1 26	15 — 15	FINAL REMARKS

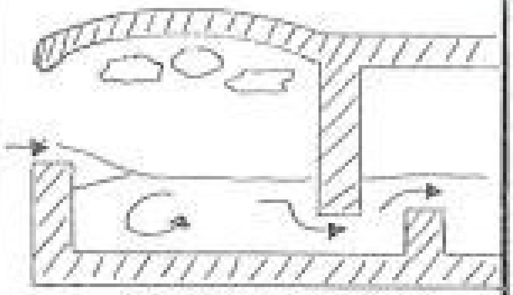
- THE 3D PARABOLIC AND PARTIALLY-PARABOLIC FLOWS ARE AN IMPORTANT SUB-CLASS FOR ENGINEERING AND THE ENVIRONMENT.
- THE SIMPLE METHOD HAS OPENED THEM TO NUMERICAL ANALYSIS WITH GREAT SUCCESS; HOWEVER, IT IS LIKELY TO BE SUPERSEDED BY SIMPLER, SIMPLEST, ETC.
- FOR SUPERSONIC FLOWS, ITERATIVE INTEGRATION CAN BE DISPENSED WITH; FOR PRESSURE EFFECTS CANNOT PENETRATE UPSTREAM.

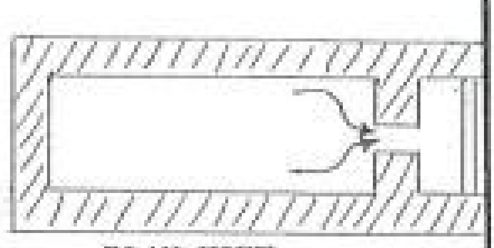
HTE 1 27	1/15	LECTURE 27. 3D ELLIPTIC AND 4D PARABOLIC PROBLEMS.

- CONTENTS:
  - 3D ELLIPTIC
    - THE GLASS TANK
    - THE ALUMINUM SMELTER
  - 4D PARABOLIC
    - THE SPARK-IGNITION ENGINE
    - THE DIESEL ENGINE
- NOTES:
  - THE MAJORITY OF PRACTICAL PROBLEMS FALL IN THIS CLASS
  - ATTEMPTS TO EMPLOY 2D REPRESENTATIONS RAISE MORE QUESTIONS THAN THEY ANSWER.

HTE 1 28	2/15	3D ELLIPTIC PROBLEMS; THE GLASS TANK, 1; THE PROBLEM.

- PURPOSE: TO MELT THE "BATCH" (SAND, SODA, ETC.): TO PRODUCE UNIFORMITY; TO REMOVE BUBBLES.
- DESIGN VARIATIONS:
- HEAT-INFLOW DISTRIBUTION;
- TANK SHAPE;
- TANK INSULATION;
- STIRRING BY BUBBLE INJECTION;
- BATCH INGREDIENTS.


VERTICAL SECTION


PLAN VIEW

HTE 1 27	3/15	THE GLASS TANK, 2; APPRECIATION OF THE MATHEMATICAL PROBLEM OF PREDICTION.

- DIMENSIONALITY: PROCESS IS 3D, BECAUSE OF:
  - CONTRACTION AT TANK OUTLET;
  - NON-UNIFORMITIES OF HEAT INPUT;
  - HEAT LOSS AND FRICTION AT WALLS.
- TRANSIENCE: PROCESS CAN (JUST) BE TREATED AS STEADY; BUT CYCLIC HEATING MAY SOMETIMES INDUCE SPECIAL EFFECTS.
- HYDRODYNAMIC PROCESSES: • FLOW IS LAMINAR;
  - VISCOSITY IS NON-UNIFORM (TEMPERATURE-DEPENDENT);
  - BUOYANCY IS ANOTHER THERMO-HYDRODYNAMIC LINK.
- HEAT TRANSFER: • BY RADIATION FROM FLAME;
  - BY CONDUCTION, RADIATION, CONVECTION IN GLASS;
  - BY CONDUCTION THROUGH WALLS.
- CHEMICAL REACTION: BATCH → GLASS.

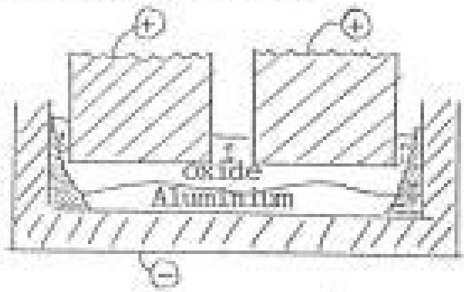
HTE 1 27	4/15	THE GLASS TANK, 3; A MATHEMATICAL MODEL.

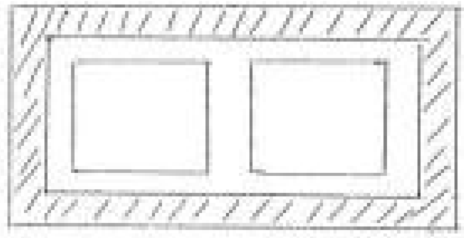
- GRID: • CARTESIAN, WITH EXTENSION INTO BRICK;
  - INTERNAL PROTRUSIONS HANDLED AS BLOCKAGES.
- FINITE-DIFFERENCE FORMULATION:
  - FULLY IMPLICIT
  - RADIATION HANDLED BY EITHER 6-FLUX MODEL (VARIOUS WAVE BANDS) OR CONDUCTION APPROXIMATION.
- SOLUTION PROCEDURE: • SIMPLE, SIMPLER OR SIMPLEST.
  - HIGHLY ELLIPTIC CHARACTER PRECLUDES PARTIALLY-PARABOLIC SIMPLIFICATION.
- POTENTIAL DIFFICULTIES: • BUOYANCY LINKAGE BETWEEN EQUATIONS MAY LEAD TO NUMERICAL INSTABILITY. • RADIATION-ENTHALPY LINK MAY SLOW CONVERGENCE (USE PEA). • 3D CHARACTER REQUIRES GRAPHICAL-OUTPUT ATTACHMENTS.

HTE 1 27	5/15	3D ELLIPTIC PROBLEMS; THE ALUMINIUM SMELTER, 1; THE PROBLEM.

- PURPOSE:
  - TO ELECTROLYSE THE OXIDE, AND SO TO FORM Al AND O_2 OR CO_2;
  - TO MAXIMISE OUTPUT.
- DESIGN VARIABLES:
  - SMELTER DIMENSIONS;
  - METAL DEPTH;
  - CURRENT;
  - CONDUCTOR CONFIGURATION.
- CRITERIA:
  - AVOID SHORT-CIRCUITING.
  - AVOID ELECTRODE EROSION.

vertical section





plan view

HTE 1 27	6/15	THE ALUMINIUM SMELTER, 2; APPRECIATION OF THE MATHEMATICAL PROBLEM OF PREDICTION.

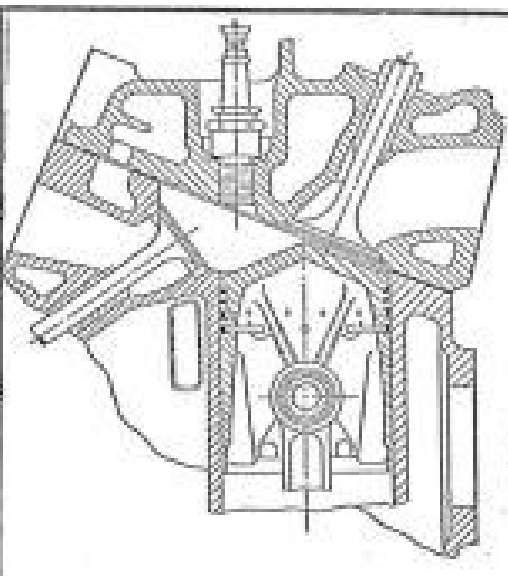
- DIMENSIONALITY: PROCESS IS 3D; BUT SHALLOWNESS RENDERS SOME TERMS UNIMPORTANT, E.G. HORIZONTAL DIFFUSION.
- TRANSIENCE: PROCESS IS NOMINALLY STEADY; BUT WAVE INSTABILITIES MAY DEVELOP.
- HYDRODYNAMIC PROCESSES: • FLOW IS TURBULENT; • IT IS DRIVEN BY ELECTROMAGNETIC FORCES; • GRAVITY OPERATES ON THE TWO FLUIDS DIFFERENTLY.
- ELECTROMAGNETIC: CURRENT AND VOLTAGE FIELDS REQUIRE SIMULTANEOUS SOLUTION.
- GEOMETRY:
  - ANODE AND FLUID SURFACES ARE NOT PLANE;
  - "FREEZE" FORMS.
  - GAS IS FORMED AT THE ANODE AND MUST FLOW AWAY.

HTE 1 27	7/15	THE ALUMINIUM SMELTER, 3; A MATHEMATICAL MODEL.

- GRID: • CARTESIAN, BUT STRETCHING IN VERTICAL DIRECTION TO ACCOMMODATE INTERFACE (N.B. GEOMETRIC CHANGES NOW FEATURE IN ITERATION LOOP). • FREEZE HANDLED BY BLOCKAGES.
- FINITE-DIFFERENCE FORMULATION: • FULLY IMPLICIT, USED ALSO FOR CHARGE CONSERVATION.
- SOLUTION PROCEDURE: • AS FOR GLASS TANK, WITH INTERFACE-MOVEMENT ADDITION.
- POTENTIAL DIFFICULTIES: • INTERFACE ADJUSTMENT COULD PRODUCE NUMERICAL INSTABILITY.
  - ELECTRO-MAGNETIC-HYDRODYNAMIC LINK COULD DO LIKEWISE (ALSO PHYSICAL INSTABILITIES MAY ARISE).
  - GAS EVOLUTION MAY REQUIRE ATTENTION.

HTE 1 27	8/15	4D PARABOLIC PROBLEMS; THE SPARK-IGNITION ENGINE, 1; THE PROBLEM.

- PURPOSE: • TO EFFECT COMPLETE COMBUSTION OF FUEL AND AIR NEAR "TOP-DEAD-CENTRE".
- DESIGN VARIABLES:
  - COMBUSTION-CHAMBER AND PISTON-CROWN SHAPE;
  - SPARK-PLUG LOCATION, ENERGY, MOMENT OF DISCHARGE;
  - VALVE SIZE, LOCATION, UPSTREAM DUCT SHAPE;
  - CHAMBER-WALL TEMPERATURE;
  - FUEL-AIR RATIO.



HTE 1 27	9/15	THE SPARK-IGNITION ENGINE, 2; APPRECIATION OF THE MATHEMATICAL PROBLEM OF PREDICTION.

- DIMENSIONALITY: • PROCESS IS 3D; AND GEOMETRY IS COMPLEX.
- TRANSIENCE: • PROCESS IS ESSENTIALLY UNSTEADY.
- HYDRODYNAMIC FEATURES: • FLOW IS TURBULENT. • FLUID IS COMPRESSIBLE, BUT MACH NUMBERS ARE LOW.
- CHEMICAL FEATURES: • SPARK CAUSES A THIN FLAME TO PASS THROUGH GAS; • "KNOCK" (SPONTANEOUS IGNITION) CAN OCCUR BEFORE THE FLAME ARRIVES.
- MOVING DOMAIN BOUNDARIES REQUIRE ATTENTION; SHOULD THE GRID BE FIXED? OR SQUASHED AND STRETCHED?

HTE 1 27	10/15	THE SPARK-IGNITION ENGINE, 3; A MATHEMATICAL MODEL.

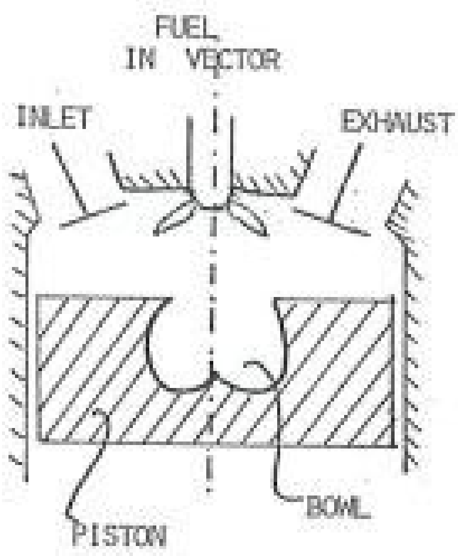
- GRID: POLAR, SQUASHING AND STRETCHING WITH BLOCKAGE (NOTE: CHAMBER SHAPE OF PANEL 8 REPRESENTS EXTREME DIFFICULTY; MANY CHAMBER SHAPES ARE EASIER).
- FINITE-DIFFERENCE FORMULATION: FULLY-IMPLICIT IS STILL THE BEST, ALLOWING TIME STEP TO BE FREELY CHOSEN.
- SOLUTION PROCEDURE: AS ABOVE.
- HANDLING OF FLAME : TREAT AS DISCONTINUITY, MOVING RELATIVE TO GAS AT EXTERNALLY PRESCRIBED RATE.
- CHEMICAL-KINETIC MODEL: MANY SPECIES MUST BE SOLVED FOR IF POLLUTANTS ARE TO BE PREDICTED, AND FOR PREDICTION OF KNOCK AND QUENCHING NEAR WALLS.

HTE 1 27	11/15	THE SPARK ENGINE, 4; CURRENT STATUS.

- ACHIEVEMENTS: ALL COMPONENTS OF THE MODEL HAVE BEEN SEPARATELY CONSTRUCTED AND TESTED, VIZ:
  - 3D, SQUASHING-STRETCHING GRID;
  - FLAME-TRAVEL TREATMENTS;
  - MULTIPLE KINETICS;
  - TURBULENCE;
  - COMPRESSIBILITY.
- THE TASKS:
  (1) TO PUT ALL FEATURES TOGETHER INTO A COMPUTER MODEL WHICH IS SUFFICIENTLY ECONOMICAL FOR PRACTICAL USE.
  (2) TO PERFORM COMPUTATIONS WHICH CAN BE COMPARED WITH EXPERIMENTAL MEASUREMENTS.
  (3) TO PERSUADE DESIGNERS TO USE THE METHOD.

HTE 1 27	12/15	4D PARABOLIC PROBLEMS; THE DIESEL ENGINE, 1; THE PROBLEM

- PURPOSE: AS FOR GASOLINE ENGINE; UTILISATION OF LARGE PROPORTION OF AIR, WITHOUT SMOKE, IS NECESSITY.
- DESIGN VARIABLES:
  - AS FOR SPARK-IGNITION.
  - FUEL-INJECTION DIRECTION, VELOCITY, DROPLET SIZE, TIMING.
- REMARK:
  TWO-PHASE PROCESSES ARE PRESENT.



HTE 1 27	13/15	THE DIESEL ENGINE, 2; APPRECIATION OF THE MATHEMATICAL PROBLEM OF PREDICTION.

- DIMENSIONALITY: • ESSENTIALLY 3D, BECAUSE OF DISCRETE-JET INJECTION OF FUEL, AND OF VALVE LOCATION. BOWL MAY BE OFF-CENTRE.
- TRANSIENCE: • ESSENTIALLY UNSTEADY.
- HYDRODYNAMIC FEATURES: • AS FOR GASOLINE; BUT FUEL-JET INJECTION IS LIKELY TO HAVE DOMINANT EFFECT.
- CHEMICAL FEATURES: • THERE IS NO RECOGNISABLE FLAME PROPAGATION; • AFTER KINETICALLY-INFLUENCED IGNITION PROCESS, COMBUSTION MAY BE MIXING-CONTROLLED.
- TWO-PHASE ASPECTS: • IF DROPLET-VAPORISATION TIMES ARE LARGE, DROPLET-SIZE DISTRIBUTIONS MUST BE COMPUTED.

HTE 1 27	14/15	THE DIESEL ENGINE, 3; A MATHEMATICAL MODEL.

- GRID: • POLAR, SQUASHING AND STRETCHING. (NOTE: UPPER-CHAMBER AND IN-BOWL REGIONS WILL NOT SQUASH). • FINE GRID WILL BE NEEDED IN FUEL-JET REGION. • BLOCKAGES USED.
- FINITE-DIFFERENCE FORMULATION: • AS ABOVE.
- SOLUTION PROCEDURE: • AS ABOVE (NOTE THAT THE TEMPERATURE EQUATION MAY AS WELL BE SOLVED FOR THE PISTON).
- POTENTIAL DIFFICULTIES: • REPRESENTATION OF THE FUEL-JET IMPINGEMENT ON METAL SURFACES. • HANDLING FINE DETAILS OF GEOMETRY (E.G. "LIP RADIUS" OF BOWL), WITH A GRID WHICH IS NOT PROHIBITIVELY EXPENSIVE.

HTE 1 27	15/15	FINAL REMARKS.

- THE SCIENTIFIC BASIS OF NUMERICAL FLUID MECHANICS, HEAT AND MASS TRANSFER, AND CHEMICAL REACTION, IS NOW QUITE SECURE.
- AN ADEQUATE MATHEMATICAL BASIS EXISTS; BUT THERE IS STILL MUCH TO BE LEARNED ABOUT OPTIMUM SOLUTION ALGORITHMS.
- COMPUTER PROGRAMS EXIST WHICH ARE QUITE SERVICEABLE; BUT MUCH IMPROVEMENT IS NEEDED, ESPECIALLY IN RESPECT OF ECONOMY, FLEXIBILITY, ACCESSIBILITY.
- UNTIL NOW, ALMOST ALL ATTENTION HAS BEEN CONCENTRATED UPON SINGLE-PHASE FLOW. METHODS FOR MULTI-PHASE FLOWS HAVE HOWEVER NOW BEEN DEVELOPED, AND ARE BEGINNING TO BE USED.

REFERENCES

AMSDEN A A & HARLOW F H (1970)

"The SMAC Method".

Los Alamos Scientific Laboratory, Report No. LA-4370.

CARETTO L S, GOSMAN A D, PATANKAR S V & SPALDING D B (1973)

"Two calculation procedures for steady, three-dimensional flows with recirculation".

Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics, Vol. II, pp.60-68.

Published by Springer-Verlag, Heidelberg.
Edited by J Ehlers, K Hepp, H A Weidenmüller.

COURANT R, ISAACSON E & REES M (1952)

"On the solution of non-linear hyperbolic differential equations by finite differences".

Communications on Pure & Applied Mathematics, 5, p 243.

ELGHOBASHI S E & PUN W M (1974)

"A theoretical and experimental study of turbulent diffusion flames in cylindrical furnaces".

Proc. 15th Symposium on Combustion, see also Imperial College, London, Mechanical Engineering Department Report No. HTS/74/15.

GOSMAN A D & LOCKWOOD F C (1973a)

"Incorporation of a flux model for radiation into a finite-difference procedure for furnace calculations".

14th Int. Symp. on Combustion, The Combustion Institute, Pittsburgh, pp 661-671.

GOSMAN A D & LOCKWOOD F C (1973b)

"Predictions of the influence of turbulent fluctuations on flow and heat transfer in furnaces".

Imperial College, London, Mechanical Engineering Department Report No. HTS/73/52.

GOSMAN A D & PUN W M (1974)

"The TEACH Program".

HTS Course Notes, Imperial College, London, Mechanical Engineering Department.

GOSMAN A D, PUN W M, RUNCHAL A K, SPALDING D B & WOLFSHTEIN M (1969)

"Heat and mass transfer in recirculating flows".

Academic Press, London.

GOSMAN A D & SPALDING D B (1970)

"Computation of laminar flow between shrouded rotating discs".

Imperial College, London, Mechanical Engineering Department Report No. EF/TN/A/30.

HAMAKER H C (1947)

"Radiation and heat conduction in light-scattering material".

Philips Res. Rep. 2, pp 55-67, p 103, 112, 420.

HARLOW F H (1969)

"Numerical methods for fluid dynamics; an annotated bibliography".

Los Alamos Laboratory Report LA-4281.

HARLOW F H (1973)

"Turbulence transport modelling".

AIAA Selected Reprint Series, Vol. XIV, AIAA, New York.

HARLOW F H & NAKAYAMA P I (1968)

"Transport of turbulence energy decay rate".

Los Alamos Sci. Lab., University of California, LA-3854.

HARLOW F H & WELCH J E (1965)

"Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface".

Physics of Fluids, Vol. 8, no. 12, pp 2182-2189.

HOTTEL H C & SAROFIM A F

"The Effect of Gas Flow Patterns on Radiative Transfer in Cylindrical Furnaces"

International Journal of Heat and Mass Transfer vol 8 pp 1153-1169. Pergamon Press 1965.

KOLMOGOROV A N (1942)

"Equations of turbulent motion of an incompressible fluid".

Izv Akad Nauk SSSR Ser Phys, Vol 6, no 1/2, pp 56-58.

(Translated into English at Imperial College, London, Mechanical Engineering Department Report No. ON/6 (1968)).

LAUNDER B E & SPALDING D B (1972)

"Mathematical models of turbulence".

Academic Press, London & New York.

LAUNDER B E & SPALDING D B (1974)

"The numerical computation of turbulent flows".

Computer Methods in App. Mechanics & Engg., Vol. 3, pp 269-289.

NG K H & SPALDING D B (1972)

"Turbulence model for boundary layers near walls".

The Physics of Fluids, Vol. 15, no. 1, pp 20-30.

PATANKAR S V & SPALDING D B (1972)

"A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows".

International Journal of Heat & Mass Transfer, Vol. 15, pp 1787-1806, Pergamon Press.

PRANDTL L (1945)

"Uber ein neues Formelsystem für die ausgebildete Turbulenz".

Nachr Akad der Wissenschaft in Göttingen, Göttingen: van den Loeck und Ruprecht, pp 6-19.

PUN W M & SPALDING D B (1968)

"A procedure for predicting the velocity and temperature distributions in a confined, steady, turbulent, diffusion flame".

XVIII International Astronautical Congress, Pergamon Press/ PWN Polish Scientific Publishers, pp 3-21.

PUN W M & SPALDING D B (1976)

"A general computer program for two-dimensional elliptic flows".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/76/2.

RODI W (1972)

"The prediction of free turbulent boundary layers by use of a two-equation model of turbulence".

PhD Thesis, University of London.
Imperial College, London, Mechanical Engineering Dept. Report No. HTS/72/24.

RODI W & SPALDING D B (1970)

"A two-parameter model of turbulence, and its application to free jets".

Wärme und Stoffübertragung, Band 3, p 85-95.

Imperial College, London, Mechanical Engineering Dept. Report No. BL/TN/B/12.

RUNCHAL A K & WOLFSHTEIN M (1966)

"A finite-difference procedure for the integration of the Navier-Stokes equations".

Imperial College, London, Mechanical Engineering Dept. Report No. SF/TN/1.

SAFFMAN P G (1970)

"A model for inhomogeneous turbulent flow".

Proc. of the Royal Society, London, Vol. A317, pp 417-433.

SCHLICHTING H (1960)

"Boundary-layer theory".

4th Ed., McGraw Hill, New York.

SCHUSTER A (1905)

Astrophysics Journal.

Vol. 21, pp 1-22.

SPALDING D B (1967)

"Heat transfer from turbulent separated flows".

Journal of Fluid Mechanics, Vol. 27, part 1, pp 97-109.

SPALDING D B (1969)

"The prediction of two-dimensional steady, turbulent elliptic flows".

International Seminar on Heat & Mass Transfer in Flows with Separated Regions, Herceg-Novi, Yugoslavia.

Imperial College, London, Mechanical Engineering Dept. Report No. EF/TN/A/16.

SPALDING D B (1972a)

"A two-equation model of turbulence".

Commemorative Lecture for Prof. F. Bosnajakovic, VDI-Forschungsheft, Vol.549, pp 5-16.

SPALDING D B (1972b)

"Mathematical models of free turbulent flows".

Monograph Instituto Nazionale di Alta Matematica, Mathematics Symposium, Vol. IX, pp 391-416.

SPALDING D B (1973a)

"Heat and mass transfer in aircraft propulsion".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/73/55.

SPALDING D B (1973b)

"Heat and mass transfer in the environment".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/73/56.

SPALDING D B (1974c)

"The calculation of convective heat transfer in complex flow systems".

Proc. 5th International Heat Transfer Conf., published by The Science Council of Japan, Vol. VI, pp 44-60.

SPALDING D B (1975a)

"Transfer of heat in rivers, bays, lakes and estuaries, and in the atmosphere - THIRBLE".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/75/4.

SPALDING D B (1975c)

"GENMIX: A general computer program for two-dimensional parabolic phenomena".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/75/17.

SPALDING D B (1976a)

"Basic equations of fluid mechanics and heat and mass transfer analysis of convective flows".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/76/6.

SPALDING D B (1976b)

"Turbulence models; a lecture course".

Imperial College, London, Mechanical Engineering Dept. Report No. HTS/76/17.

TENNEKES H & LUMLEY J (1972)

"A first course in turbulence".

MIT Press, U.S.A.

INDEX TO APPENDICES:

APPENDIX 'A'

A Finite-Difference Procedure for Solving The Equations of the Two-Dimensional Boundary Layer.

by

S V Patankar and D B Spalding

Int. J. Heat Mass Transfer vol 10, pp 1389-1411. Pergamon Press Ltd 1967

Int. J. Heat Mass Transfer. Vol. 10, pp. 1389-1415. Pergamon Press Ltd. 1967. Printed in Great Britain

A FINITE-DIFFERENCE PROCEDURE FOR SOLVING THE EQUATIONS OF THE TWO-DIMENSIONAL BOUNDARY LAYER

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Abstract—A general, implicit, numerical, marching procedure is presented for the solution of parabolic partial differential equations, with special reference to those of the boundary layer. The main novelty lies in the choice of a grid which adjusts its width so as to conform to the thickness of the layer in which significant property gradients are present. The non-dimensional stream function is employed as the independent variable across the layer.

The capabilities of the method are demonstrated by application to: the heated flat plate in a high-Mach-number laminar stream; the axi-symmetrical turbulent jet in moving and stagnant surroundings; and the radial turbulent wall jet.

NOMENCLATURE

(The number in the parentheses denotes the equation of first mention.)

- a, a group of symbols in the convection term (3.1.1);
- A, a coefficient in the difference equation (3.2.3);
- b, a group of symbols in the convection term (3.1.1);
- B, a coefficient in the difference equation (3.2.3);
- c, a group of symbols in the diffusion term (3.1.1);
- $\bar{c}_f$, mean skin-friction coefficient;
- C, a quantity in the difference equation (3.2.3);
- d, diameter of the jet nozzle (4.2.1);
- D, a coefficient in the difference equation for V_1 (3.2.4);
- D_k, dissipation of the turbulent kinetic energy (2.1.9);
- E, a coefficient in the transformed difference equation (3.3.7);
- f, a fraction between zero and unity;
- F, a quantity in the transformed difference equation (3.3.7);
- g_1, g_2, g_3, g_4, coefficients in the difference form of the convection term (3.2.2);
- G_1, $\equiv \rho V_1$ (2.1.2);
- G_2, $\equiv \rho V_2$ (2.1.2);
- G_ϕ, $\equiv \rho V_\phi$ (2.1.4);
- h, specific enthalpy (2.2.6);
- $\tilde{h}$, stagnation enthalpy (2.1.7);
- H, a coefficient in the transformed difference equation (3.3.7);
- J, diffusional flux (3.4.1);
- k, mean kinetic energy of the fluctuating motion per unit mass (2.1.7);
- l_1, l_2, the length scales in direction 1 and 2 (2.1.1);
- l_m, mixing length (2.2.3);
- l_k, length scale associated with k (2.2.4);
- L, a coefficient in the transformed difference equation (3.3.11);
- m_j, mass fraction of a chemical species j (2.1.8);

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M, a quantity in the transformed difference equation (3.3.11);
N, number of strips across the layer;
p, pressure (2.1.4);
P, a coefficient in the transformed difference equation (3.3.2);
Q, a quantity in the transformed difference equation (3.3.2);
r, radius, distance from the axis of symmetry (2.1.2);
$r_{\frac{1}{2}}$, the radius at which the velocity is one-half of the centre-line velocity;
R, a coefficient in the transformed difference equation (3.3.5);
Re, Reynolds number, $(\rho_0 u_0 x/\mu_0)$;
R_j, rate of generation of the chemical species j (2.1.8);
s, distance along the $\xi_1 \sim \xi_2$ plane (2.1.1);
S_ϕ, Stanton-number function for ϕ (2.3.1);
St, the Stanton number;
T, absolute temperature (4.1.1);
u, velocity in longitudinal direction (4.2.1);
u^+, dimensionless velocity $[u/\sqrt{(\tau_S/\rho)}]$ (4.3.2);
u_c, velocity at the wall-jet slot;
u_{max}, maximum velocity in the wall-jet profile;
V_1, velocity in direction 1 (2.1.3);
V_2, velocity in direction 2 (2.1.3);
V_r, velocity in radial direction (2.1.6);
V_θ, velocity in direction θ (2.1.4);
x, longitudinal distance (4.2.1);
y, distance across the layer;
y^+, dimensionless distance $(y\sqrt{(\tau_S\rho)}/\mu)$ (4.3.2);
$y_{\frac{1}{2}}$, half-value thickness of the wall jet;
y_c, thickness of the wall-jet slot;
y_b, characteristic thickness of the layer used to calculate the mixing length (4.3.1).

Greek symbols

β, the angle made by direction 1 with the symmetry axis (2.1.4);
γ, the ratio of specific heats;
κ, a constant in the mixing-length formula (4.3.1);
λ_0, a constant in the the mixing-length formula (4.3.1);
μ, laminar viscosity (2.3.1);
$\mu_{1,eff}$, effective viscosity in direction 1 (2.1.4);
$\mu_{\theta,eff}$, effective viscosity in direction θ;
ξ_1, coordinate in direction 1 (2.1.1);
ξ_2, coordinate in direction 2 (2.1.1);
ρ, fluid density (2.1.3);
σ, laminar Prandtl or Schmidt number (2.3.1);
$\sigma_{h,eff}$, effective Prandtl number (2.1.7);
$\sigma_{j,eff}$, effective Schmidt number for species j (2.1.8);
$\sigma_{k,eff}$, effective Prandtl number for the diffusion of k (2.1.7);
$\sigma_{\theta,eff}$, $= \mu_{1,eff}/\mu_{\theta,eff}$ (2.1.6);
τ_S, shear stress at the wall;
ϕ, a typical dependent variable (2.1.10);
ϕ^*, a predetermined value of ϕ used in the grid-control formula (3.6.3);
Φ, a term representing generation of ϕ in the typical equation (2.1.10);
ψ, a stream function (2.1.5);
ω, dimensionless steam function, coordinate in direction 2 (2.1.11).

Subscripts

D, the downstream point on a portion of the grid;
$D+, D-$, points near to and at the same value of ξ_1 as D;
E, the external boundary of the layer;
G, a free-stream boundary;
$G-$, a point within the layer, near to and at the same value of ξ_1 as G;
I, the internal boundary of the layer;
S, a wall boundary;
$S+$, a point within the layer, near to and at the same value of ξ_1 as S;
U, the upstream point on a portion of the grid;
$U+, U-$, points near to and at the same value of ξ_1 as U;

- 0, the initial line; jet-nozzle condition;
- 1, the coordinate direction 1;
- 2, the coordinate direction 2;
- θ, the direction perpendicular to the radius and in a plane normal to the symmetry axis;
- ϕ, pertaining to the dependent variable ϕ.

1. INTRODUCTION

1.1. *The problem considered*

HEAT-, mass- and momentum-transfer in steadily flowing media are governed by elliptic differential equations. Because these are difficult to solve, the elliptic equations are often, and legitimately, truncated to a parabolic form; these truncated equations are the boundary-layer equations.

The present paper provides a new method of solving these equations. That a new method may be desirable is shown by the fact that existing methods are still not widely used; they are either too expensive to operate, too difficult to adapt to particular problems, or too prone to failures and inaccuracies. For this reason many authors, including the present ones, have put forward approximate procedures of calculation [1], in which only a few, integral, forms of the partial differential equations are solved; but these too have their shortcomings.

The solution procedures which are simplest in concept are those of the numerical, finite-difference type. Many variants have been suggested and employed successfully; but they are open to the above-mentioned objections. The new method is also of the finite-difference variety; but it embodies special devices for reducing the computation time, without loss of accuracy, and for bringing many types of problems within the scope of a single computer programme.

1.2. *Some remarks on earlier finite-difference methods*

Classification. Finite-difference procedures for parabolic equations can be distinguished according to the co-ordinate systems which they employ, and according to whether they are "explicit" or "implicit" in character.

Choice of finite-difference formula. For unsteady-heat-conduction problems, the explicit methods are typified by the Binder–Schmidt procedure [2], whereas the implicit methods are typified by that of Crank and Nicholson [3]. The advantages and disadvantages are well known. Explicit methods involve only simple arithmetic; but the time interval must not exceed a fixed proportion of the square of the space interval divided by the thermal diffusivity. Implicit methods involve much more arithmetic per time interval, because simultaneous equations appear, requiring solution by matrix-inversion or successive-substitution techniques; on the other hand, they are free from any limitation on the size of the time interval.

Whether explicit or implicit methods are preferable for heat-conduction problems remains a matter of opinion. For the problems which arise in boundary-layer theory, on the other hand, the superiority of the implicit method is becoming widely recognised. This superiority results from the fact that the explicit method here has an upper limit on the distance interval in the stream direction; and this limit is directly proportional to the fluid velocity. Since this velocity may become very small near a wall, very small distance steps must be taken; the implicit method, which is free from this restriction, therefore requires much less computing time than the explicit one.

Although the implicit method necessitates matrix inversion, the matrix is a simple one; so inversion may be achieved by way of recurrence relations. The procedure of Pashkonov [4] is typical; it employs the Crank–Nicholson form of the finite-difference formulae, and has been developed for predicting the flow in laminar boundary layers.

Choice of coordinate system. Figure 1 illustrates a typical choice of coordinate grid, and enables its disadvantage to be clearly observed. The $x \sim y$ grid is rectangular, and coincides

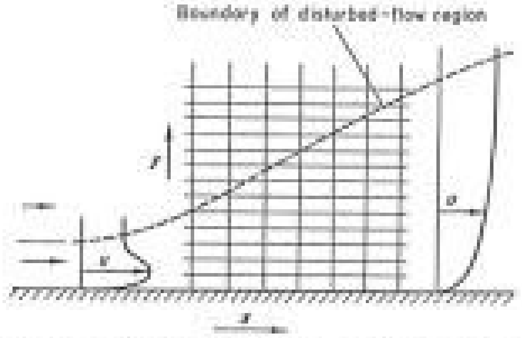

FIG. 1. The growth of a wall jet into a surrounding stream. The sketch shows how the region of disturbed flow widens in the downstream direction.

at one edge with the wall which bounds the region of interest. The other boundary of this region, shown dotted on the diagram, extends obliquely across the grid. Now, to achieve accuracy, a certain minimum number of grid points should be contained at the upstream end within the thickness of the layer. Obviously therefore, sufficient accuracy in the upstream region is purchased at the expense of an excessively fine grid for the downstream region. So a rectangular grid is likely to be inefficient; computations made with its aid are unnecessarily expensive.

Although several means have been proposed for solving this difficulty, none is both neat and generally applicable. There is therefore a need for a general coordinate system which allows the requirements of accuracy to be reconciled with those of elegance and of computational efficiency.

1.3. *Outline of the present contribution*

The calculation procedure that is described below is of the "implicit" variety. The scheme differs slightly from that of Crank and Nicholson; but, like that method, it allows the grid spacing in the main-stream direction to be freely chosen.

A greater innovation is the choice of cross-stream variable; for this we adopt the *non-dimensional* stream function, ω, defined so that ω always equals zero at one edge of the boundary layer and unity at the other. The procedure combines the advantages of stream-line coordinates with those of restricting the boundary layer to a finite domain.

Real boundary layers seldom have observable "edges"; so those which are used to normalize the stream function are artificial; but they may be freely chosen; and we have devised a method of choosing them, during the course of the integration procedure, which ensures computational efficiency.

Although the method is a general one for parabolic equations, it is here illustrated by reference to equations having particular physical significance, i.e. to those expressing the laws of conservation of momentum, material, and energy (of various kinds). These equations are assembled, and expressed in the appropriate coordinate system, in Section 2; there we also introduce certain auxiliary relations which are appropriate to turbulent flow; and the main features of the grid-control technique are described in sub-section 2.4. The procedure of numerical solution is described in Section 3; its use is illustrated in Section 4, by calculations of three phenomena: a laminar boundary layer, a free turbulent flow, and a turbulent wall jet.

2. THE EQUATIONS OF THE BOUNDARY LAYER

2.1. *The partial differential equations for axi-symmetrical flow*

The coordinate system. Figure 2 illustrates the coordinate system which will be adopted for the axi-symmetrical flow to which attention will be confined.† The coordinate directions 1 and 2 are orthogonal, or nearly so; the values of the coordinates are ξ_1 and ξ_2, so defined that the element of distance ds in a plane through the axis of symmetry is given by:

$$ds = \sqrt{[(l_1\, d\xi_1)^2 + (l_2\, d\xi_2)^2]}. \qquad (2.1.1)$$

The length scales l_1 and l_2 remain to be defined.

† Plane flows are, of course, members of the axi-symmetrical family.

The direction of the constant-ξ_2 lines is chosen so that, for the most part, it is nearly parallel to the local direction of the component of the velocity vector in the plane of the diagram.

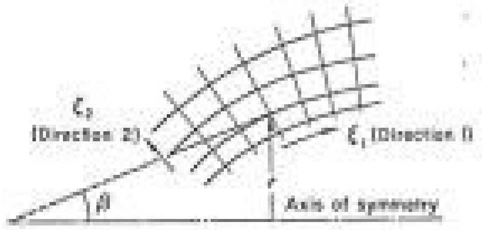

FIG. 2. Co-ordinate system for axi-symmetrical flow.

The constant-ξ_2 lines make the angle β with the symmetry axis; the angle is of course, in general, a function of ξ_1 and ξ_2. The constant-ξ_1 lines are, correspondingly, everywhere almost perpendicular to stream lines. It will be supposed, as part of the boundary-layer approximation, that the heat-conduction, diffusion and viscous-action vectors have significant components only in direction 2.

The equations. We start from the following forms of the differential equations expressing the main conservation laws of steady flow. The symbols employed are defined in the Nomenclature.

Mass conservation:

$$\frac{\partial}{\partial \xi_1}(r l_2 G_1) + \frac{\partial}{\partial \xi_2}(r l_1 G_2) = 0, \qquad (2.1.2)$$

or, alternatively:

$$\rho V_1 = G_1 = \frac{1}{r l_2}\frac{\partial \psi}{\partial \xi_2},$$

$$\rho V_2 = G_2 = \frac{-1}{r l_1}\frac{\partial \psi}{\partial \xi_1}. \qquad (2.1.3)$$

Momentum conservation in direction 1:

$$\frac{G_1}{l_1}\frac{\partial V_1}{\partial \xi_1} + \frac{G_2}{l_2}\frac{\partial V_1}{\partial \xi_2} = \frac{1}{r l_1 l_2}\frac{\partial}{\partial \xi_2}\left(\frac{r l_1}{l_2}\mu_{1,\mathrm{eff}}\frac{\partial V_1}{\partial \xi_2}\right) - \frac{1}{l_1}\frac{\partial p}{\partial \xi_1} + V_2 G_1 \frac{\partial \beta}{\partial \xi_1} + \frac{V_\theta G_\theta}{r}\sin\beta. \qquad (2.1.4)$$

Momentum conservation in direction 2:

$$0 = \frac{-1}{l_2}\frac{\partial p}{\partial \xi_2} - \frac{V_1 G_1}{l_1}\frac{\partial \beta}{\partial \xi_1} + \frac{V_\theta G_\theta}{r}\cos\beta. \qquad (2.1.5)$$

Momentum conservation in direction θ:

$$\frac{G_1}{l_1}\frac{\partial V_\theta}{\partial \xi_1} + \frac{G_2}{l_2}\frac{\partial V_\theta}{\partial \xi_2} = \frac{1}{r l_1 l_2}\frac{\partial}{\partial \xi_2} \times \left(r^2 \frac{l_1}{l_2}\frac{\mu_{1,\mathrm{eff}}}{\sigma_{\theta,\mathrm{eff}}}\frac{\partial (V_\theta/r)}{\partial \xi_2}\right) - \frac{V_r G_\theta}{r}. \qquad (2.1.6)$$

Equation for stagnation enthalpy, $\bar{h}$:

$$\frac{G_1}{l_1}\frac{\partial \bar{h}}{\partial \xi_1} + \frac{G_2}{l_2}\frac{\partial \bar{h}}{\partial \xi_2} = \frac{1}{r l_1 l_2}\frac{\partial}{\partial \xi_2}\left[\frac{r l_1}{l_2}\mu_{1,\mathrm{eff}} \times \left\{\frac{1}{\sigma_{h,\mathrm{eff}}}\frac{\partial \bar{h}}{\partial \xi_2} + \left(\frac{1}{\sigma_{k,\mathrm{eff}}} - \frac{1}{\sigma_{h,\mathrm{eff}}}\right)\frac{\partial k}{\partial \xi_2} + \left(1 - \frac{1}{\sigma_{h,\mathrm{eff}}}\right)\frac{\partial}{\partial \xi_2}\left(\frac{V_1^2}{2}\right) + \left(\frac{1}{\sigma_{\theta,\mathrm{eff}}} - \frac{1}{\sigma_{h,\mathrm{eff}}}\right)\frac{\partial}{\partial \xi_2}\left(\frac{V_\theta^2}{2}\right) - \frac{1}{\sigma_{\theta,\mathrm{eff}}} \times \frac{l_2^2}{r l_1} V_\theta^2 \cos\beta\right\}\right]. \qquad (2.1.7)$$

Conservation of chemical species j:

$$\frac{G_1}{l_1}\frac{\partial m_j}{\partial \xi_1} + \frac{G_2}{l_2}\frac{\partial m_j}{\partial \xi_2} = \frac{1}{r l_1 l_2}\frac{\partial}{\partial \xi_2}\left(\frac{r l_1}{l_2}\frac{\mu_{1,\mathrm{eff}}}{\sigma_{j,\mathrm{eff}}}\frac{\partial m_j}{\partial \xi_2}\right) + R_j. \qquad (2.1.8)$$

Conservation of kinetic energy of turbulence, k:

$$\frac{G_1}{l_1}\frac{\partial k}{\partial \xi_1} + \frac{G_2}{l_2}\frac{\partial k}{\partial \xi_2} = \frac{1}{r l_1 l_2}\frac{\partial}{\partial \xi_2} \times \left(\frac{r l_1}{l_2}\frac{\mu_{1,\mathrm{eff}}}{\sigma_{k,\mathrm{eff}}}\frac{\partial k}{\partial \xi_2}\right) + \mu_{1,\mathrm{eff}}\left\{\left(\frac{1}{l_2}\frac{\partial V_1}{\partial \xi_2}\right)^2 + \frac{1}{\sigma_{\theta,\mathrm{eff}}}\left[\frac{r}{l_2}\frac{\partial (V_\theta/r)}{\partial \xi_2}\right]^2\right\} - D_k. \qquad (2.1.9)$$

All of these equations, except (2.1.2), (2.1.3) and (2.1.5), can be regarded as possessing the common form:

$$\frac{G_1}{l_1}\frac{\partial\phi}{\partial\xi_1} + \frac{G_2}{l_2}\frac{\partial\phi}{\partial\xi_2} = \frac{1}{l_1 l_2 r}\frac{\partial}{\partial\xi_2} \times \left(\frac{r l_1}{l_2}\frac{\mu_{1,\text{eff}}}{\sigma_{\phi,\text{eff}}}\frac{\partial\phi}{\partial\xi_2}\right) + \Phi. \qquad (2.1.10)$$

Here ϕ stands for any of the dependent variables: V_1, V_θ, $\tilde{h}$, m_j, k; and Φ stands for terms appearing on the right-hand side which do not contain $\partial\phi/\partial\xi_2$. Specifically, the meanings of Φ can be expressed by the following table.

Table 1. Significances of Φ

When ϕ stands for:	Φ stands for:
V_1	$\frac{-1}{l_1}\frac{\partial p}{\partial\xi_1} + V_1 G_1 \frac{\partial\beta}{\partial\xi_2} + \frac{V_\theta G_\theta}{r}\sin\beta$
V_θ	$\frac{-1}{r l_1 l_2}\frac{\partial}{\partial\xi_2}\left(l_1 \frac{\mu_{1,\text{eff}}}{\sigma_{\theta,\text{eff}}} V_\theta \cos\beta\right) - \frac{V_1 G_\theta}{r}$
$\tilde{h}$	$\frac{1}{r l_1 l_2}\frac{\partial}{\partial\xi_2}\left[\frac{r l_1}{l_2}\mu_{1,\text{eff}}\left\{\left(\frac{1}{\sigma_{h,\text{eff}}} - \frac{1}{\sigma_{k,\text{eff}}}\right)\frac{\partial k}{\partial\xi_2} + \left(1 - \frac{1}{\sigma_{h,\text{eff}}}\right)\frac{\partial}{\partial\xi_2} \times \left(\frac{V_1^2}{2}\right) + \left(\frac{1}{\sigma_{h,\text{eff}}} - \frac{1}{\sigma_{\theta,\text{eff}}}\right) \times \frac{\partial}{\partial\xi_2}\left(\frac{V_\theta^2}{2}\right) - \frac{1}{\sigma_{\theta,\text{eff}}}\frac{l_2}{r l_1} V_\theta^2 \cos\beta\right\}\right]$
m_j	R_j
k	$\mu_{1,\text{eff}}\left\{\left(\frac{1}{l_2}\frac{\partial V_1}{\partial\xi_2}\right)^2 + \frac{1}{\sigma_{\theta,\text{eff}}}\left(\frac{r}{l_2}\frac{\partial(V_\theta/r)}{\partial\xi_2}\right)^2\right\} - D_k$

The similarity of their equations allows a common treatment for the variables V_1, V_θ, $\tilde{h}$, m_j and k. The equations expressing conservation of mass and of direction-2 momentum will be handled differently.

The transformation to ω as cross-stream variable. As yet l_1 and l_2 have not been defined. We now make a choice of l_2 which gives ξ_2 the significance of the non-dimensional stream function, for which we adopt the special symbol ω; thus

$$\xi_2 = \omega = \frac{\psi - \psi_I}{\psi_E - \psi_I}. \qquad (2.1.11)$$

Here ψ_I and ψ_E are the values of ψ prevailing at the internal (I) and external† (E) boundaries of the region which is to be considered; they are functions of ξ_1 which may be selected arbitrarily, but which we shall try to choose so that all the important variations in the dependent variables take place at ψ values between ψ_I and ψ_E.

Equation (2.1.11) implies:

$$\frac{\partial\psi}{\partial\xi_1} = (1 - \omega)\frac{d\psi_I}{d\xi_1} + \omega\frac{d\psi_E}{d\xi_1}, \qquad (2.1.12)$$

and:

$$\frac{\partial\psi}{\partial\xi_2} = \frac{\partial\psi}{\partial\omega} = \psi_E - \psi_I. \qquad (2.1.13)$$

These relations may be substituted into the two parts of equation (2.1.3), which expresses the mass-conservation principle. The first part gives the required relation for l_2, while the second gives an expression for G_2. Thus:

$$l_2 = \frac{\psi_E - \psi_I}{r G_1}, \qquad (2.1.14)$$

and:

$$G_2 = \frac{-1}{r l_1}\left\{(1 - \omega)\frac{d\psi_I}{d\xi_1} + \omega\frac{d\psi_E}{d\xi_1}\right\}. \qquad (2.1.15)$$

Substitution of these two results into the general differential equation (2.1.10) yields the

† The labels "internal" and "external" are most apt for axi-symmetrical coordinate systems for which ξ_2 increases with distance in the radial direction. However, being only labels, they can be used generally also.

following new form of this equation:

$$\frac{\partial \phi}{\partial \xi_1} - \frac{1}{(\psi_E - \psi_I)} \left\{ (1-\omega)\frac{d\psi_I}{d\xi_1} + \omega \frac{d\psi_E}{d\xi_1} \right\} \frac{\partial \phi}{\partial \omega} = \frac{1}{(\psi_E - \psi_I)^2} \frac{\partial}{\partial \omega} \left(\frac{r^2 l_1 G_1 \mu_{1,\mathrm{eff}}}{\sigma_{\phi,\mathrm{eff}}} \frac{\partial \phi}{\partial \omega} \right) + \Phi \frac{l_1}{G_1} \qquad (2.1.16)$$

It is this equation which forms the starting point for the finite-difference procedure.

The direction-2 momentum equation. Equation (2.1.5) can be integrated along a line of constant ξ_1 to give:

$$p - p_I = \int_{\xi_{2,I}}^{\xi_2} \left(-\frac{V_1 G_1}{l_1} \frac{l_2 \partial \beta}{\partial \xi_1} + \frac{V_\theta G_\theta}{r} l_2 \cos \beta \right) d\xi_2. \qquad (2.1.17)$$

Substitution from equation (2.1.14), with ω written in place of ξ_2, yields:

$$p - p_I = (\psi_E - \psi_I) \int_0^{\omega} \left(\frac{-V_1}{l_1 r} \frac{\partial \beta}{\partial \xi_1} + \frac{V_\theta G_\theta}{r^2 G_1} \cos \beta \right) d\omega. \qquad (2.1.18)$$

This equation must also be used, in general, during the finite-difference solution procedure. When, however, $\partial\beta/\partial\xi_1$ is negligible, as is often the case, and when the rotational velocity V_θ is negligible, equation (2.1.18) reduces simply to: $p = p_I = p_E$; the pressure can be taken as uniform across the boundary layer.

2.2. *Auxiliary relations*

Geometrical relations. Appearing in equations (2.1.16) and (2.1.18) are the geometrical quantities: r, l_1 and β. It is necessary to calculate these as functions of the independent variables ξ_1 and ω. An example will suffice to show how these calculations can be made.

Suppose that the boundary-layer region is bounded by a solid surface of rotation, as shown in Fig. 3. Then the direction 1 can be taken as parallel to this surface and the direction 2 (ω-direction) as normal to it. Now suppose that, along the surface, where ω equals zero, l_1 equals unity; then ξ_1 stands for the distance along the surface.

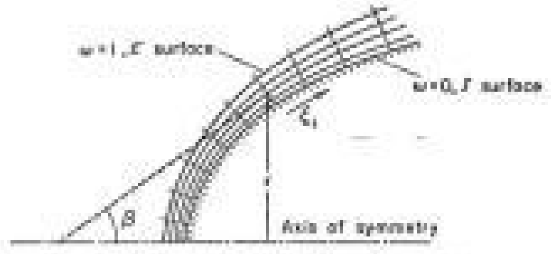

FIG. 3. Illustration of a typical coordinate system.

If constant-ξ_1 lines are normals to the surface, and if the coordinate system can be treated as orthogonal, β is a function of ξ_1 alone; moreover the function is prescribed by the shape of the surface, which we may suppose to be known. A consequence is that the length-scale factor l_1 is given by:

$$\xi_1 = \text{fixed}: l_1 = 1 - \left(\frac{\partial \beta}{\partial \xi_1} \right) \int_0^{\omega} l_2 \, d\omega, \qquad (2.2.1)$$

where l_2, of course, varies with ω in accordance with equation (2.1.14) and the velocity distribution.

The radius r is also calculable from simple geometrical considerations. The relevant formula is:

$$\xi_1 = \text{fixed}: \quad r = r_I + (\cos \beta) \int_0^{\omega} l_2 \, d\omega. \qquad (2.2.2)$$

The quantity $\int_0^{\omega} l_2 \, d\omega$, it should be understood, equals the distance from the wall measured along a line of constant ξ_1.

In the case illustrated in Fig. 2, the decision to locate the I-surface along the wall requires no justification; but it should be recognized that it is a free decision, not a forced one. Where to place the E-surface is not at first obvious; we can repeat only that we want the region between the two surfaces to contain all the points at which viscous, diffusion and heat-conduction effects are significant. How this can be ensured will be described in Section 2.4 below.

Exchange coefficients. Equations are needed which will connect $\mu_{1,\text{eff}}$, and the exchange-coefficient ratios $\sigma_{\theta,\text{eff}}$, $\sigma_{h,\text{eff}}$, $\sigma_{k,\text{eff}}$ and $\sigma_{j,\text{eff}}$, with the dependent variables of the calculation.

If the flow is laminar, $\mu_{1,\text{eff}}$ is the laminar viscosity; standard data sources allow this to be connected quantitatively with the enthalpy and composition of the fluid. Then also $\sigma_{\theta,\text{eff}}$ equals unity, because the laminar viscosity is isotropic; and $\sigma_{k,\text{eff}}$ becomes unimportant because k is zero. The quantities $\sigma_{h,\text{eff}}$ and $\sigma_{j,\text{eff}}$ are respectively the Prandtl number and the Schmidt number, on which standard data sources once again give information.†

When the flow is turbulent, different relationships are appropriate. Usually each σ_{eff} is taken as uniform in the turbulent region, and approximately equal to unity; the main interest centres on the calculation of $\mu_{1,\text{eff}}$.

One formula in common use is that of Prandtl [5]. This may be written as:

$$\mu_{1,\text{eff}} = \frac{l_m^2\rho}{l_2}\left|\frac{\partial V_1}{\partial \xi_2}\right| \qquad (2.2.3)$$

where l_m is the so-called mixing length. The latter quantity is usually taken as dependent on the distance from the wall; $\int_0^\omega l_2\, d\omega$; examples are given below (Section 4).

Another formula, proposed by Kolmogorov [6] and Prandtl [7] connects the viscosity with the kinetic energy of turbulence, and another length scale, l_k. This has been used by Monin [8] and Glushko [9], among others. The formula may be expressed as:

$$\mu_{1,\text{eff}} = l_k\rho k^{1/2}. \qquad (2.2.4)$$

The length scale l_k is also taken as some function of distance from the wall.

Particularly when V_θ is of the same order of magnitude as V_1, more elaborate formulae for $\mu_{1,\text{eff}}$ are needed to express experimental findings. All known proposals could be easily incorporated into the solution procedure that is to be described; there is therefore no necessity to introduce further examples.

The dissipation-rate. Since the employment of equation (2.2.4) necessitates solution of the differential equation for the kinetic energy of turbulence, equation (2.1.9), it is appropriate to mention that the dissipation-rate D_k must enter an auxiliary relation. An example, in accordance with dimensional analysis, is;

$$D_k = \text{constant} \cdot \rho k^{3/2}/l_k. \qquad (2.2.5)$$

Here l_k is of course the same as the quantity appearing in equation (2.2.4).

Equations (2.2.4) and (2.2.5) are recommended, it should be added, only where the flow is fully turbulent. This condition can be expressed in terms of a "local Reynolds number of turbulence": $l_k\rho k^{1/2}/\mu$ should be very much greater than unity. What functions are appropriate when the condition is not fulfilled is not at present clear.

Thermodynamic relationships. The dependent variables are linked by many relations which express either, definitions, or thermodynamic laws, or material-property relationships. Among these is:

$$\tilde{h} = h + \frac{V_1^2}{2} + \frac{V_\theta^2}{2} + k, \qquad (2.2.6)$$

where h is the specific enthalpy; the kinetic energy associated with direction-2 motion is neglected. Equation (2.2.6) allows the enthalpy to be calculated from the values of $\tilde{h}$, V_1, V_θ and k appearing in the solutions of the differential equations. If the equation for k is not being solved for use in a viscosity relation like (2.2.4),

† Strictly speaking, in laminar flow, $\sigma_{h,\text{eff}}$ reduces to the Prandtl number only under certain conditions (e.g. equal specific heats for all components, and no chemical reaction; or, uniform composition). There is, however, no point in discussing the exceptions here.

it is usual to drop this equation and neglect the contributions of k to $\bar{h}$.

Other important properties linked to enthalpy and concentration by thermodynamic relations are the temperature T, and the density ρ. The equations are too well known to require presentation.

2.3. *Initial and boundary conditions*

The domain of integration. The solutions of the equations are to be confined to the region: $\xi_1 \geq \xi_{1,0}$, $0 \leq \omega \leq 1$. Figure 4 illustrates this. There is no need to specify the right-hand edge of the domain.

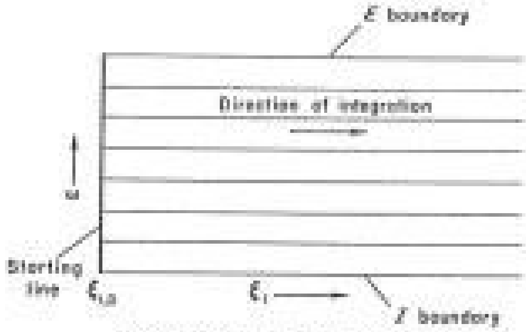

FIG. 4. The domain of integration.

In order to integrate the parabolic equations, it is necessary to know values of all the variables along the "starting line" where ξ_1 equals $\xi_{1,0}$; these values comprise the initial conditions. We shall suppose that they are always available.

Equally necessary is information about conditions at the I and E boundaries. This information may be in the form of prescribed distributions of the values of the variables along these lines; alternatively, values of gradients, or other functions, of the variables may be prescribed. Some special kinds of boundary condition will be mentioned in the present section.

The pressure should be prescribed along a single boundary; for only one integration constant is required for equation (2.1.18). Two types of prescription will be mentioned below.

Wall-flux laws for turbulent flow. When the boundary layer is turbulent and a solid wall is present, the region near the wall exhibits very steep gradients of velocity, and often of other variables too. Since the velocity is also low there, the $\partial\phi/\partial\xi_1$ term is locally negligible in the differential equation;† consequently the variation of ϕ can be calculated by reference to the remaining terms, which involve differential coefficients with respect to ω alone. Thus a "Couette-flow analysis" gives a good approximation to the exact solution of the equation.

It is possible, but somewhat wasteful, to crowd together the constant-ω grid lines in the region near a wall so as to perform this Couette-flow analysis at each step of the finite-difference solution procedure. But it is also possible, and more economical, to carry out Couette-flow analyses, once for all, before the particular finite-difference calculation is started; the results of these analyses can then be incorporated into algebraic relationships which serve as boundary conditions. The economy arises from the resulting freedom to have the constant-ω lines more evenly spaced; it suffices, for example, to have one which lies somewhat beyond the outer edge of the "laminar sub-layer", and to connect values of variables and fluxes there, to values and fluxes at the wall, by way of algebraic formulae. A simple example of such a formula is presented in Section 4.3 below. More elaborate formulae will be found in [10]. They have the general form:

$$\frac{\left(\frac{\mu}{\sigma_\phi l_2}\frac{\partial\phi}{\partial\omega}\right)_S}{(\phi_{S+} - \phi_S)\,G_{1,S+}} = S_{\phi,S}\left(\frac{G_{1,S+}\int_0^{\omega_{S+}} l_2\,d\omega}{\mu_S}, \sigma_{\phi,S}, \ldots\right). \qquad (2.3.1)$$

† This fact is best understood by reference to equation (2.1.4), with l_1 and l_2 put equal to unity; it is disguised in equation (2.1.16) by the fact that the definition of ω contains G_1. The fact is well known to laminar- boundary-layer specialists, and easily proved.

Here the subscript S denotes the solid-surface boundary, either E or I, and $S+$ denotes the nearest grid point to S; μ is the laminar viscosity of the fluid; and σ_ϕ is the laminar Prandtl or Schmidt number appropriate to property ϕ. The numerator of the left-hand side has the significance of the flux of ϕ across the boundary; the function $S_{\phi,S}$ is a Stanton number, having as its main argument the Reynolds number

$$(G_{1,S+} \int_0^{\omega_{S+}} l_2 \, d\omega/\mu_S)$$

and the laminar Prandtl or Schmidt number. Other arguments may account for the presence of mass transfer, roughness, pressure gradient and property variations; but they involve quantities appropriate to points S and $S+$ alone.

When the flux through the wall is given, equation (2.3.1) is used for the calculation of ϕ_S; when, on the other hand, ϕ_S is given, the flux through the wall is calculated. These remarks apply whether ϕ stands for temperature, enthalpy, concentration, velocity, or any other entity for which equation (2.1.10) is valid; of course, different words are used to describe the various cases; for example, the "Stanton number of momentum transfer" is better known as "one half of the drag coefficient".

Other conditions at the boundaries. When the I or E boundary does not coincide with a wall, it is usual for the values of the dependent variables to be prescribed there; for example, if the variable is k, the kinetic energy of the fluctuating motion, its value on the boundary will ordinarily be that which prevails in the stream to which the domain of integration is adjacent.

If the boundary coincides with the symmetry axis, the gradient of ϕ with respect to normal distance must be zero. This fact can serve as a boundary condition in appropriate circumstances, for example when the centre-line of an axi-symmetrical jet is in question.

The prescription of the pressure. It has already been mentioned that an equation exists, namely (2.1.18), from which the pressure at any point along the constant-ξ_1 line can be calculated whenever *one* pressure is prescribed; this could be either p_I or p_E. Often this prescription is given through the velocity at a boundary, coupled with the statement that pressure, velocity and density are linked there by the Euler equation, i.e. by equation (2.1.4), with the shear-stress term neglected. This situation usually arises in external-flow situations, for example boundary layers on aerofoils, and free jets.

When the flow is an internal one, like that in a diffuser for example, neither pressure nor velocity can be calculated directly from input data. Instead, the continuity equation must be solved for the whole flow; this gives an additional condition to be satisfied by the velocity and density profiles at the next step of the integration; it must be solved simultaneously with equations which represent the finite-difference form of the differential equation. Further discussion will be deferred until these equations have been introduced, in Section 3.3 below.

2.4. *The choice of ψ_I and ψ_E*

The purpose. ψ_I and ψ_E, it will be remembered, are functions of ξ_1 which we are still free to specify as we wish. The requirements are: that the constant-ξ_1 lines will be approximately normal to stream lines, at any rate in the regions of highest velocity; and that the region $0 \leq \omega \leq 1$ contains all points having significant ϕ gradients.

Because, in boundary layers, gradients are finite only in *slender* regions, for which the long dimension is roughly parallel to the flow direction, fulfilment of the second requirement satisfies the first one also. We shall now describe some suitable procedures for controlling ψ_I and ψ_E.

The symmetry axis as a boundary. If the region in which gradients are significant encloses the symmetry axis, as in the case of a wake behind a cylinder in longitudinal flow, the choice for ψ_I is obvious; it should be placed equal to a constant, for example zero.

A solid wall as a boundary. When the region containing significant gradients extends right up to a solid wall, as in the case of the flow in a diffuser, the objective can be achieved by making one of the boundaries coincide with the wall. Let us use the subscript S once more to denote this boundary. If the wall is impermeable, ψ_S must be a constant, which can be arbitrarily fixed; if it is permeable, however, ψ_S must vary in accordance with equation (2.1.15), which reduces (for ω equal to zero or unity as appropriate) to:

$$G_{2,S} = \frac{-1}{rl_1}\frac{\mathrm{d}\psi_S}{\mathrm{d}\xi_1}. \tag{2.4.1}$$

In some cases, $G_{2,S}$, the mass-transfer rate across the wall, is fixed by the data of the problem; this occurs, for example, when suction of the boundary layer through the wall is effected by external means. In other cases, as when sublimation occurs from the solid into the gas at a rate controlled by heat transfer, $G_{2,S}$ has to be calculated at each stage from the local values of some of the ϕ's. Always, however, a differential equation is obtained for ψ_S; this can be solved, by the usual numerical techniques, during the course of the integration.

When the boundary is free. The last two choices for boundary-ψ values were so straightforward that they merited discussion only to serve as contrasts to that which now confronts us: the choice of the value of the stream function along the boundary separating the region of interest from an adjoining region of the flow in which the gradients are negligible. This boundary might be the outer "edge" of an axi-symmetrical turbulent jet, injected into a moving stream; the outer "edge" of the laminar boundary layer on a flat plate is another example. We shall use the subscript G to denote such a boundary.

Two cases of this kind must now be distinguished. In the first, a definite edge to the boundary layer can be established without arbitrariness; this case arises when the flow is turbulent and may be assumed to obey the Prandtl 1925 mixing-length hypothesis [5]; for then, as may be seen from [11] for example, the transport properties all vanish along a surface which is not infinitely remote from the region. Of course, this vanishing applies only when the laminar contribution to the transport properties is already being neglected. Since this neglect is justified only where the turbulent component is *large*, the case may be regarded as rather artificial; nevertheless, it is simple, useful, and sufficiently accurate for most purposes.

In the second case of a free boundary, the transport properties neither vanish, nor fall to a small fraction of their values elsewhere, along a definite boundary line. This is true of laminar flows, and of turbulent ones which are supposed to obey the Kolmogorov–Prandtl [6, 7] postulate, for example, and for which the free-stream turbulence level may not be neglected. In this case the G boundary is more arbitrary.

Free-stream boundary with vanishing transport properties. When $\mu_{1,\mathrm{eff}}$ (say) vanishes along the G boundary, a differential equation for ψ_G can be obtained from the general partial differential equation (2.1.16). Just outside the G boundary, $\partial\phi/\partial\omega$ is zero; the equation therefore reduces to:

$$\left(\frac{\partial\phi}{\partial\xi_1}\right)_G = \left(\phi\frac{l_1}{G_1}\right)_G. \tag{2.4.2}$$

Consideration of a point just inside the G boundary, for which $\partial\phi/\partial\xi_1$ and $(\Phi l_1/G_1)$ cannot be significantly different, therefore leads (whether ω equals zero or unity at the boundary designated by subscript G) to:

$$-\frac{\mathrm{d}\psi_G}{\mathrm{d}\xi_1} = \frac{1}{(\psi_E - \psi_I)} \lim_{\omega \to \omega_G} \left[\frac{\dfrac{\partial}{\partial\omega}\left(\dfrac{r^2 l_1 G_1 \mu_{1,\mathrm{eff}}}{\sigma_{\phi,\mathrm{eff}}}\cdot\dfrac{\partial\phi}{\partial\omega}\right)}{\dfrac{\partial\phi}{\partial\omega}}\right]. \tag{2.4.3}$$

Here we see, incidentally, an implied test discriminating between the two cases. If $\mu_{1,\text{eff}}$ is proportional to $\partial\phi/\partial\omega$, the limit will be finite (because $r^2 l_1 G_1/\sigma_{\phi,\text{eff}}$ is finite); then a real boundary can indeed exist. Otherwise the limit does not converge; $d\psi_G/d\xi_1$ becomes infinite; ψ_G must be infinite; so *all* the fluid must be contained within the range $0 \leqslant \omega \leqslant 1$.

When the limit converges, which is true when Prandtl's 1925 mixing-length hypothesis is used, equation (2.4.3) provides a satisfactory specification of $d\psi_G/d\xi_1$. With its aid, the ψ_G values along the boundaries can be calculated during the course of integration, just as in the case of ψ_S. These calculations of the boundary values of ψ achieve the desired effect of causing the coordinate grid to expand and contract so as always to meet the requirement for computational efficiency.

Free-stream boundary with non-vanishing transport properties. Since the limit in equation (2.4.3) does not converge unless the transport property vanishes, we shall apply now the full partial differential equation (2.1.16) just inside the G boundary with a special consideration for evaluating the $\partial\phi/\partial\xi_1$ term: we shall seek to locate the boundary so that, on the grid line just inside the G boundary, the value of ϕ will be equal to a predetermined number ϕ^*. This number can be chosen, for example, so that the difference $(\phi_G - \phi^*)$ is a certain small percentage of the maximum ϕ-difference across the layer. Now the value of $\partial\phi/\partial\xi_1$ can be calculated along the grid line just inside the G boundary from the current known value of ϕ and the value of ϕ^* desired at the downstream station. The finite-difference formula for this will be given in Section 3.6 below. It is sufficient to note here that, if the value of ϕ^* is properly chosen, we can be sure that the grid will always conform to the region in which significant gradients of ϕ are present.

The above procedures have been outlined as ones which seem best at present. However, it should be remembered that the "entrainment rate" $d\psi_G/d\xi_1$ is in any case arbitrary; its sole justification is computational efficiency. It is, therefore, permissible to abandon the above procedures at any time that it becomes convenient to do so; for example, one may put a maximum limit on the entrainment rate to avoid excessive curvature of the streamlines near the boundary.

2.5. *Closure to section*

Now that formulae have been indicated with the aid of which the boundary-ψ's can be calculated, the whole mathematical structure has been outlined. It remains to show how the equations can be solved; this is the function of the following sections. The finite-difference procedures are explained in Section 3, while Section 4 demonstrates their utility by way of examples.

3. THE RECOMMENDED FINITE-DIFFERENCE PROCEDURE

3.1. *Outline*

For convenience, let us express equation (2.1.16) as:

$$\frac{\partial\phi}{\partial\xi_1} + (a + b\omega)\frac{\partial\phi}{\partial\omega} = \frac{\partial}{\partial\omega}\left(c\frac{\partial\phi}{\partial\omega}\right) + \phi\frac{l_1}{G_1}, \tag{3.1.1}$$

where

$$a = -\frac{1}{(\psi_E - \psi_I)}\frac{d\psi_I}{d\xi_1}, \tag{3.1.2}$$

$$b = -\frac{1}{(\psi_E - \psi_I)}\frac{d(\psi_E - \psi_I)}{d\xi_1}, \tag{3.1.3}$$

and

$$c = \frac{G_1 r^2 l_1 \mu_{1,\text{eff}}}{(\psi_E - \psi_I)^2\,\sigma_{\phi,\text{eff}}}. \tag{3.1.4}$$

We shall solve equations of this type by step-by-step forward integration. Therefore, at every step in the integration, the values of ϕ will be known at discrete values of ω and at one value of ξ_1; our task will be to obtain the values of

ϕ at the same values of ω, but at a downstream value of ξ_1. By repetition of this basic operation, the whole field of interest can be covered.

The discrete values of ω and ξ_1, which are decided beforehand, define a grid; a portion of this is shown in Fig. 5. Points U and D represent respectively the upstream and downstream points at a given ω; points at nearby values of ω will be called $U+$, $U-$, $D+$, $D-$. The

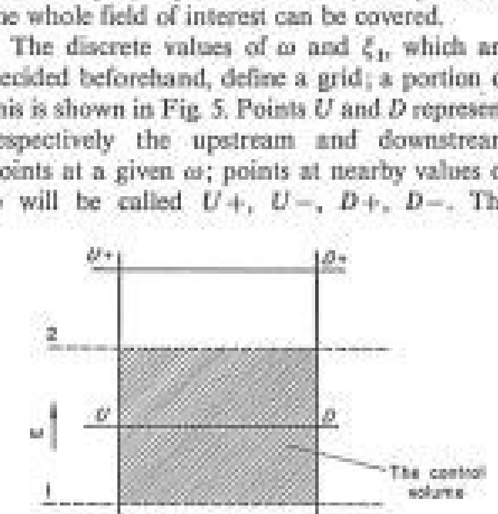

FIG. 5. Location of points referred to in the difference equation.

dashed lines 1 and 2 are the lines of constant ω, midway between $UU-$ and $UU+$ respectively. Lines 1 and 2 form, together with the two lines of constant ξ_1, a control volume (shown shaded) which will be useful for expressing the convection terms.

We shall describe in Section 3.2 below how equation (3.1.1) can be put in finite-difference form. Our representation of the convection terms, i.e. the terms on the left-hand side of equation (3.1.1), is based on an integrated average over a small control volume. This implies that the convection to point D is influenced by the values of ϕ at all the neighbouring points; it thus increases stability. Also, the integral equation over the whole layer is then automatically satisfied. While expressing the second-order term $\partial/\partial\omega(c\,\partial\phi/\partial\omega)$, we need to decide the value of ξ_1 at which this term will be evaluated. In general, we can use:

$$f\left[\frac{\partial}{\partial\omega}\left(c\frac{\partial\phi}{\partial\omega}\right)\right]_U + (1-f)\left[\frac{\partial}{\partial\omega}\left(c\frac{\partial\phi}{\partial\omega}\right)\right]_D,$$

where f is a number between zero and unity and subscripts U and D denote locations of evaluation. When f is unity, this form reduces to that of the explicit method, which, as mentioned earlier, has severe limitations on the step-length $(\xi_{1,D} - \xi_{1,U})$. For any value of f different from unity, the scheme becomes implicit. It can be shown, at least in simple cases, that instability is avoided if $0 \leqslant f \leqslant 0{\cdot}5$. The case of $f = 0{\cdot}5$ corresponds to the method of Crank and Nicholson [3]. We have decided to take the value of f as zero, as this combines stability with convenience. In other words, we shall evaluate the second-order term along the line $\xi_1 = \xi_{1,D}$.

3.2. *The difference formulae*

We shall now express the various terms in equation (3.1.1) by finite-difference formulae.

The convective terms. The terms on the left-hand side of equation (3.1.1) can be expressed as:

$$\frac{\displaystyle\int_{\xi_{1,U}}^{\xi_{1,D}}\int_{\omega_1}^{\omega_2}\left\{\frac{\partial\phi}{\partial\xi_1} + (a+b\omega)\frac{\partial\phi}{\partial\omega}\right\}\mathrm{d}\omega\,\mathrm{d}\xi_1}{(\omega_2-\omega_1)(\xi_{1,D}-\xi_{1,U})}.$$

Now if we assume that ϕ varies linearly between the grid points in both ξ_1 and ω directions, it is easy to express the above double integral in terms of the values of ϕ at U, $U+$, $U-$, D, $D+$ and $D-$. The resulting expression can be written as:

$$g_1\,\phi_{D+} + g_2\,\phi_D + g_3\,\phi_{D-} + g_4,$$

where the g's are obtainable in terms of known quantities, including the values of ϕ at $\xi_1 = \xi_{1,U}$. The detailed expressions for the g's will not be given here; they can be easily obtained by straightforward algebra.

The flux term. As mentioned in Section 3.1, the second-order term $\partial/\partial\omega(c\,\partial\phi/\partial\omega)$, representing the diffusional flux, will be evaluated along the line $\xi_1 = \xi_{1,D}$. However, in order that the resulting difference equations become linear, we shall evaluate the *coefficient* c along $\xi_1 = \xi_{1,U}$ where all the quantities are known. Thus the finite-difference form of $\partial/\partial\omega(c\,\partial\phi/\partial\omega)$ will be:

$$\frac{2}{\omega_{D+} - \omega_{D-}}\left\{\frac{(c_{U+} + c_U)}{2}\frac{(\phi_{D+} - \phi_D)}{(\omega_{D+} - \omega_D)} - \frac{(c_U + c_{U-})}{2}\frac{(\phi_D - \phi_{D-})}{(\omega_D - \omega_{D-})}\right\}.$$

The alternative practice would be to evaluate the c's along $\xi_1 = \xi_{1,D}$; but then the solution of the resulting non-linear equations would need iteration. We shall not consider this possibility here.

The source term. Finally, we need to express the term $\Phi\, l_1/G_1$ in finite-difference form. The simplest procedure would be to evaluate this term from the known quantities at $\xi_{1,U}$. A better practice† is to express the term $\Phi\, l_1/G_1$ as:

$$(\Phi\, l_1/G_1)_U + \left[\frac{\partial}{\partial\phi}(\Phi\, l_1/G_1)\right]_U (\phi_D - \phi_U).$$

When ϕ stands for V_1, the pressure-gradient term $\partial p/\partial\xi_1$ appears in the corresponding expression for Φ. Since the pressure p will not always be known, this case needs special consideration. It is easy to see that the finite-difference form of $\partial p/\partial\xi_1$ will be:

$$\frac{p_D - p_U}{\xi_{1,D} - \xi_{1,U}}.$$

An addition relationship can be obtained for the unknown pressure p_D from equation (2.1.18).

† A still better one, it might appear, would be to employ a more elaborate expression which allows for the dependence of a single Φ on several different dependent variables. However, to do so would be to introduce more unknowns than can be coped with by the solution procedure which is advocated below; we therefore refrain from this elaboration.

We can write:

$$p_D - p_{D-} = (\psi_E - \psi_I)\int_{\omega_{D-}}^{\omega_D}\left(-\frac{V_1}{l_1 r}\frac{\partial\beta}{\partial\xi_1} + \frac{V_\phi G_\phi}{r^2 G_1}\cos\beta\right)d\omega. \qquad (3.2.1)$$

Although in principle it is possible to handle this complete equation, the increased algebraic complication may obscure the main elements of the procedure. Therefore, for the purposes of presentation only, we shall use a simpler form of equation (3.2.1): we shall assume that pressure p is uniform for a given value of ξ_1. This assumption is valid when the stream lines are not highly curved and the swirl velocity V_ϕ is small. Incidentally, this case happens to be the one of most practical importance. The following treatment is valid for this case. A reader interested in cases of non-uniform pressure in the ξ_2 direction can work out the full implications of equation (3.2.1) along similar lines.

The complete difference equation. So far, we have explained how the individual terms can be expressed in finite-difference form. Putting them together, we compile the complete difference equation as follows:

$$g_1\phi_{D+} + g_2\phi_D + g_3\phi_{D-} + g_4 = \frac{2}{\omega_{D+} - \omega_{D-}}\left\{\frac{(c_{U+} + c_U)}{2}\frac{(\phi_{D+} - \phi_D)}{(\omega_{D+} - \omega_D)} - \frac{(c_U + c_{U-})}{2}\frac{(\phi_D - \phi_{D-})}{(\omega_D - \omega_{D-})}\right\} + \left[\frac{\Phi\, l_1}{G_1}\right]_U + \left[\frac{\partial}{\partial\phi}(\Phi\, l_1/G_1)\right]_U(\phi_D - \phi_U). \qquad (3.2.2)$$

It is easy to see that, by rearrangement, this equation can be reduced to the form:

$$\phi_D = A\phi_{D+} + B\phi_{D-} + C, \qquad (3.2.3)$$

where A, B and C are obtainable in terms of known quantities. If the pressure p_D is given, then the equation for V_1 will also have the same form as (3.2.3); however, in the case of confined flow, the pressure p_D will appear as unknown. The form of the equation will then be:

$$V_{1,D} = AV_{1,D+} + BV_{1,D-} + C + Dp_D. \quad (3.2.4)$$

Equation (3.2.3) or (3.2.4) is the final outcome of our finite-difference formulation. There will be one such equation for every grid point except for those on the I and E boundaries. Only in certain circumstances will the above-mentioned procedure need modification. We shall describe this point in Section 3.4 below. Now we turn to the problem of solving the algebraic equations like (3.2.3).

3.3. *Solution of the difference equations*

Procedure for unconfined flows. For unconfined flows, the pressure p_D can be obtained before solving the boundary-layer equations. Then the difference equations for all the ϕ's including V_1 are of the form (3.2.3).

Let us suppose that the grid lines divide the thickness of the layer into N strips. If subscript i denotes a node corresponding to a value of ω, then equations of the type (3.2.3) can be written as:

$$\phi_i = A_i\phi_{i+1} + B_i\phi_{i-1} + C_i, \quad (3.3.1)$$

for $i = 2, 3, 4, \ldots N$. The values of ϕ_1 and ϕ_{N+1} will be given as boundary conditions. (When the gradients of ϕ at the boundaries are given, we shall modify the formulation of equation (3.2.3) so that the following solution procedure can still be used. This point will be described in Section 3.4 below.) We shall first transform equation (3.3.1) into the following form:

$$\phi_i = P_i\phi_{i+1} + Q_i, \quad (3.3.2)$$

where

$$\left.\begin{aligned} P_i &= \frac{A_i}{1 - B_iP_{i-1}}, \\ Q_i &= \frac{B_iQ_{i-1} + C_i}{1 - B_iP_{i-1}}, \\ P_2 &= A_2 \\ \text{and} \quad Q_2 &= B_2\phi_1 + C_2. \end{aligned}\right\} \quad (3.3.3)$$

After the calculation of P's and Q's, it is a simple matter to obtain ϕ's from equation (3.3.2) by successive substitution starting from ϕ_{N+1}.

Procedure for confined flows. When the flow is confined, the pressure p_D is not directly specified. On the other hand, we have an additional relationship that the rate of change of the total mass flow in the whole duct with the streamwise co-ordinate ξ_1 depends only on the mass-transfer rates at the confining walls. Since pressure p_D appears only in an equation for V_1, the equations for other ϕ's can be solved by the above procedure for unconfined flows. We therefore describe below the procedure for solving the equation for V_1, for confined flows. This is being presented here for the sake of completeness; however, it should be mentioned that we have not yet used this procedure for solving any actual problem and that the examples in Section 4 below are all of the unconfined-flow variety.

As shown in Section 3.2, the equation for V_1 has the form given by (3.2.4); we shall rewrite that equation as follows:

$$V_{1,i} = A_iV_{1,i+1} + B_iV_{1,i-1} + C_i + D_ip, \quad (3.3.4)$$

for

$$i = 2, 3, 4, \ldots N,$$

where p stands for the pressure along the line $\xi_1 = \xi_{1,D}$. This equation can be transformed into:

$$V_{1,i} = P_iV_{1,i+1} + Q_i + R_ip, \quad (3.3.5)$$

where

$$\left.\begin{aligned} P_i &= \frac{A_i}{1 - B_iP_{i-1}}, \\ Q_i &= \frac{B_iQ_{i-1} + C_i}{1 - B_iP_{i-1}}, \\ R_i &= \frac{B_iR_{i-1} + D_i}{1 - B_iP_{i-1}}; \\ P_2 &= A_2, \\ Q_2 &= B_2V_{1,1} + C_2, \\ R_2 &= D_2. \end{aligned}\right\} \quad (3.3.6)$$

We transform equation (3.3.5) once again to express all V_i's in terms of $V_{1,N+1}$, as follows:

$$V_{1,i} = E_i V_{1,N+1} + F_i + H_i p, \qquad (3.3.7)$$

where

$$\left.\begin{aligned} E_i &= P_i E_{i+1}, \\ F_i &= P_i F_{i+1} + Q_i, \\ H_i &= P_i H_{i+1} + R_i; \\ E_N &= P_N, \\ F_N &= Q_N, \\ H_N &= R_N; \\ E_{N+1} &= 1, \\ F_{N+1} &= H_{N+1} = 0. \end{aligned}\right\} \qquad (3.3.8)$$

$$p = \frac{M - \sum_{i=2}^{N} L_i(E_{i+1} + E_i) V_{1,N+1} - \sum_{i=2}^{N} L_i(F_{i+1} + F_i)}{\sum_{i=2}^{N} L_i(H_{i+1} + H_i)}. \qquad (3.3.12)$$

At this stage we shall introduce the continuity equation for the whole duct. From equation (2.1.14) we see that:

$$\int_0^1 \frac{d\omega}{G_1} = \int_0^1 \frac{r l_2 \, d\omega}{(\psi_E - \psi_I)}. \qquad (3.3.9)$$

It is easy to see that the right-hand side is calculable for any value of ξ_1; because the variation of $(\psi_E - \psi_I)$ can be obtained from the prescribed mass-transfer rates at the confining walls, and the integral

$$\int_0^1 r l_2 \, d\omega$$

is known from the geometry of the duct. We can write†:

$$\left[\int_0^1 \frac{d\omega}{G_1}\right]_D - \left[\int_0^1 \frac{d\omega}{G_1}\right]_U$$

$$\approx \sum_{i=1}^{N} \frac{(\omega_{i+1} - \omega_i)}{0.5(\rho_{i+1} + \rho_i)_U} \left\{\frac{-2}{(V_{1,i+1} + V_{1,i})^2}\right\}_U$$
$$\times \{(V_{1,i+1} + V_{1,i})_D - (V_{1,i+1} + V_{1,i})_U\}, \qquad (3.3.10)$$

which has the form:

$$\sum_{i=2}^{N} L_i (V_{1,i+1} + V_{1,i})_D = M, \qquad (3.3.11)$$

where L's and M are known quantities. Now substituting from equation (3.3.7), we get, after some re-arrangement: [equation (3.3.12) above]

Using this value of p, we can obtain the values of all V_1's from equation (3.3.7).

3.4. *Special procedures*

It has been implicitly assumed so far that the values ϕ_1 and ϕ_{N+1} at the boundaries are known. Sometimes, however, instead of the value of ϕ, the gradient of ϕ is specified along the boundary. In such cases, the difference equations for the nodes near the boundary need some modification.

The equation (3.1.1) can be written as:

$$\frac{\partial \phi}{\partial \xi_1} + (a + b\omega)\frac{\partial \phi}{\partial \omega} = \frac{\partial}{\partial \omega}(r J_\phi) + \Phi \frac{l_1}{G_1}, \qquad (3.4.1)$$

where J_ϕ stands for the flux caused by the gradient of ϕ. Now, if the point $D-$ lies on the boundary, we can write the flux term $\partial/\partial\omega\,(rJ_\phi)$ in finite-difference form as follows:

$$\frac{2}{\omega_{D+} - \omega_{D-}} \left\{\frac{(r_{D+} + r_D)}{2} \frac{(\phi_{D+} - \phi_D)}{(\omega_{D+} - \omega_D)} - \frac{(r_D + r_{D-})}{2} J_{\phi,D-}\right\},$$

† Here we neglect the density variations, so as to have only $V_{1,k}$'s as unknowns. A different practice will have to be adopted when the density variations have a significant effect on the pressure variations. For example, $\partial\rho/\partial\xi_1$ values can be stored in previous steps of the integration, and used for forward extrapolation here. It is however premature to suggest remedies for difficulties that have not yet been encountered.

where $J_{\phi,B-}$ is known from the prescribed gradient of ϕ at the boundary. This formulation avoids explicit reference to the unknown boundary value ϕ_{D-}. Of course, when all other ϕ's have been calculated, ϕ_{B-} can be deduced from the prescribed gradient and from the value of ϕ_D.

3.5. *Choice of forward step*

To perform the forward integration, the size of the step length $(\xi_{1,D} - \xi_{1,U})$ must be decided. Since the present finite-difference formulation is of implicit type, stability will be maintained even when the size of the step is large; however, for good accuracy, small steps are necessary. The most economical size of the step for a particular class of problem can be found by experience. A simple procedure is to make the step length proportional to the thickness of the layer, i.e. to put:

$$(\xi_{1,D} - \xi_{1,U}) = \text{const.} \times \frac{1}{I_1}\int_0^1 l_2 \, d\omega. \qquad (3.5.1)$$

This will be quite satisfactory for most of the turbulent boundary layers where the thickness of the layer varies approximately linearly with the longitudinal distance. For laminar boundary layers, a step length proportional to the square of the layer thickness would be more appropriate.

In some situations, the growth of the layer thickness is very slow, for example in a mixing layer between two streams of nearly equal velocities; in these cases we can choose the step length so that the extra quantity of fluid entrained during that step is equal to a definite fraction of the quantity of fluid already existing in the layer. This rule can be expressed in the following form:

$$\left[\frac{d(\psi_E - \psi_I)}{d\xi_1}\right]_U (\xi_{1,D} - \xi_{1,U}) = \text{const.} \times (\psi_E - \psi_I)_U. \qquad (3.5.2)$$

3.6. *Formula for grid control*

The quantity $(\psi_E - \psi_I)$ appears in all the difference equations given so far. It is therefore necessary to describe the means of calculating $(\psi_E - \psi_I)$ for successive values of ξ_1. It is this quantity that determines the actual size of the grid; the following formulae will therefore be called the grid-control formulae.

It is easy to see that:

$$(\psi_E - \psi_I)_D = (\psi_E - \psi_I)_U + \left[\frac{d(\psi_E - \psi_I)}{d\xi_1}\right]_U \cdot (\xi_{1,D} - \xi_{1,U}). \qquad (3.6.1)$$

where $G-$ denotes the grid point next to G on a constant $-\xi_1$ line, and the subscript $GG-$ boundary happens to be a wall or a line of symmetry, the calculation of the corresponding $d\psi/d\xi_1$ is straightforward, for example by use of equation (2.4.1). We consider below the more important case of a free boundary. Again it is necessary to distinguish between the two sub-classes of this case.

Free-stream boundary with vanishing transport properties. For this case, we write the equation (2.4.3) in finite-difference form as follows:†

$$-\frac{d\psi_G}{d\xi_1} = \frac{1}{(\psi_E - \psi_I)} \frac{(4r^2 I_1 G_1 \mu_{1,eff})_{GG-}}{|\omega_G - \omega_{G-}|}, \qquad (3.6.2)$$

where $G-$ denotes the grid point next to G on a constant $-\xi_1$ line, and the subscript $GG-$ indicates evaluation in between the points G and $G-$. The equation (3.6.2) is obtained by assuming ϕ to stand for V_1, and by taking the profile for V_1 as parabolic with distance in the interval between $G-$ and G. It is possible to devise alternative forms.

Free-stream boundary with non-vanishing transport properties. The basis for the grid-control formula for this case has been explained in Section 2.4. In order to present the formula in finite-difference form, we need to re-write the equation (2.1.16) after putting ω equal to zero or unity and by expressing the term

† $\sigma_{\phi,eff}$ is unity, when V_1 is the property in question; so it does not appear explicitly in the equation.

value ϕ^*; the expression resulting from the flux term will be taken as the same as in equation (3.6.2). Thus we have:

$$-\frac{d\psi_G}{d\xi_1} = \frac{(4r^2 I_1 G_1 \mu_{1,eff})_{GG-}}{(\psi_E - \psi_I)[\omega_G - \omega_{G-}]}$$

$$+ \frac{\left\{\left(\frac{\Phi I_1}{G_1}\right)_{G_1} - \frac{\phi^* - \phi_{G-}}{\xi_{1,D} - \xi_{1,U}}\right\}(\psi_E - \psi_I)}{2(\phi_G - \phi^*)/(\omega_G - \omega_{G-})}. \quad (3.6.3)$$

Here the $\phi \sim \omega$ profile in the interval $GG-$ is assumed to be parabolic for the purpose of calculating $(\partial\phi/\partial\omega)$.

Instability in grid control. We have used an implicit scheme for formulating the difference equations and therefore can confidently expect stability. The equation (3.6.1) for calculating $(\psi_E - \psi_I)$, however, is of the explicit type; i.e. the upstream value of the derivative $d(\psi_E - \psi_I)/d\xi_1$ is used for the whole interval. This can give rise to fluctuations in the value of the thickness of the layer, when large steps in the ξ_1 direction are used. One way to avoid these fluctuations is to use a weighted mean of the current value of the derivative and that for the previous integration. The fluctuations of the boundary are more likely to arise when the velocity in the surrounding stream is zero. This is to be expected because then if the entrainment rate happens to be rather large, the boundary has to be shifted by a very large distance to entrain the extra quantity of fluid. Use of a small but finite value of velocity in the surrounding stream will restore stability. Finally, if the size of the forward step is reasonably small, instability through grid control will not normally arise.

It should be noted that the grid-control procedure is the part of the present calculation method that most needs ingenuity and care. It would be desirable in the long run to devise a single general procedure which can be applied irrespective of whether or not the transport properties vanish at the boundary.

4. APPLICATIONS

In this section we shall demonstrate the capabilities of the calculation procedure described so far, by way of three examples. The purpose of this section is to show that the present method can be successfully used for predicting heat transfer and friction in various types of flow. Though we shall, of necessity, use physical hypotheses and make comparisons with experimental data, the emphasis is on presenting a convenient mathematical tool and not on demonstrating that the hypotheses which we have used are the best ones.

A remark regarding the change of notation will be helpful here. Having completed the presentation in terms of the general coordinate system, we can now use symbols that are simpler and more familiar. Thus we use below u for V_1, and x and y for distances in direction 1 and direction 2 respectively.

4.1. *Compressible laminar boundary layer on flat plate*

Statement of problem. To test the effectiveness of the new mathematical procedure, comparison with available exact solutions is highly desirable. Therefore, as our first example, we have chosen the flat-plate laminar boundary layer for which Van Driest [12] has presented exact numerical solutions. The problem is characterized by zero pressure gradient, no mass transfer, uniform wall temperature, uniform specific heat, uniform Prandtl number (equal to 0·75), and viscosity variation given by the Sutherland law, namely:

$$\frac{\mu}{\mu_G} = \left(\frac{T}{T_G}\right)^{\frac{3}{2}} \frac{1\text{·}505}{1 + 0\text{·}505(T_G/T)}, \quad (4.1.1)$$

where μ and T respectively stand for viscosity and absolute temperature, while the subscript G denotes conditions in the main stream. The ratio of specific heats, γ, is taken as 1·4 and the density is assumed to be inversely proportional to the absolute temperature. The task is to calculate the drag coefficient and the Stanton number for various Mach numbers and for various wall-to-mainstream temperature ratios.

Details of solution procedure. In this case the partial differential equations solved were (2.1.4) and (2.1.7). The number of grid lines across the layer was 16. The initial profiles of velocity and temperature were arbitrarily taken as linear with distance. The grid-control procedure used was, of course, that for the boundary with non-vanishing viscosity. The value of ϕ^* was taken as 0·999 u_G, where u_G is the free-stream velocity. The integration was continued until the profiles of velocity and temperature ceased to change. In this state the boundary-layer thickness becomes proportional to the square root of the longitudinal distance along the plate. This equilibrium state was achieved after about 150 integration steps and 0·2 min of IBM 7090 computer time.

Results. Figures 6 and 7 respectively show the variations of $\bar{c}_f(\sqrt{Re})$ and $St(\sqrt{Re})$ with Mach number, for various temperature ratios. The full lines represent the solutions from [12] and the points show our solution. The agreement is satisfactory. Thus the present method enables one to obtain accurate solutions of the equations for "similar" boundary layers, even though it is not specifically designed for this purpose.

4.2. *Axisymmetrical turbulent jet*

Statement of problem. As our second example, we take the problem of an axi-symmetrical turbulent jet. Figure 8 shows a jet with velocity

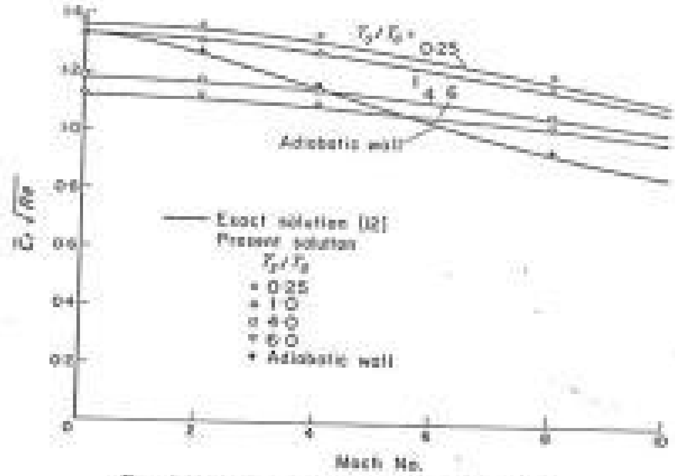

FIG. 6. Variation of mean skin-friction coefficient.

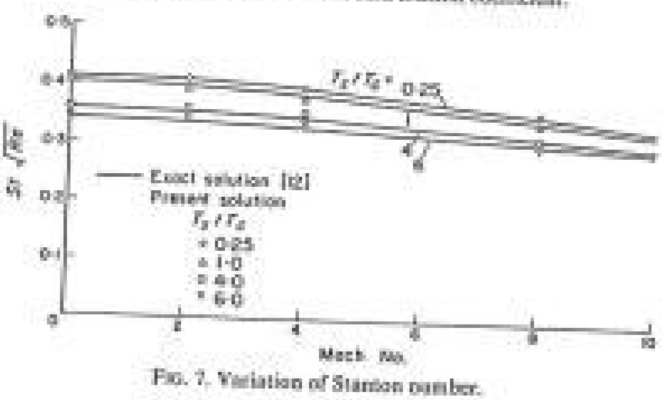

FIG. 7. Variation of Stanton number.

$u_{I,0}$ coming out from a nozzle of diameter d into a surrounding stream of uniform velocity u_E. The density is uniform. The problem is to calculate the centre-line velocity at various downstream distances, the velocity profiles, etc.

Details of solution procedure. For this case the partial differential equation (2.1.4) was solved. The effective viscosity was calculated by using Prandtl's 1925 mixing-length hypothesis, which has been described in Section 2.2. The mixing length was taken as uniform across the layer and equal to 0·0845 times a characteristic thickness of the layer, defined as the distance between two points each of which is near one of the boundaries of the layer; when the boundary coincides with a wall or with a line of symmetry, such a point lies on the boundary; when the boundary is adjacent to a free stream, the point is located such that the velocity there differs from the free-stream velocity by 1 per cent of the maximum velocity difference across the layer.

To start the integration, a linear velocity profile with a very small thickness of the layer was used. The radius of the inner boundary was calculated from the rate of entrainment from the potential core into the inner surface. After the inner radius became zero, the inner boundary was considered to be the line of symmetry. The number of grid lines across the layer was eleven. The forward step was chosen so that the extra amount of fluid entrained during each step was equal to one-tenth of the quantity of the fluid already within the layer.

Results. Figure 9 shows the decay of the centre-line velocity of the jet with downstream distance, for three velocity ratios: $u_E/u_{I,0} = 0$, 0·2, 0·5. Also shown is the line representing the relation:

$$\frac{u_I - u_E}{u_{I,0} - u_E} = \frac{6{\cdot}5}{x/d}, \qquad (4.2.1)$$

which is known to agree well with most of the experimental data for the downstream region of free jets in stagnant surroundings. The agreement with equation (4.2.1) of the present solution for $u_E/u_{I,0} = 0$ is quite good.

Each curve on Fig. 9 represents about 0·25 min of IBM 7090 computer time.

For the downstream region of a jet in stagnant surroundings, Tollmien [13] has obtained an

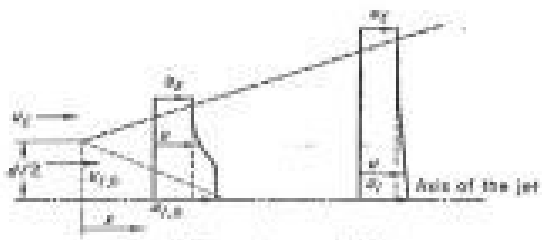

FIG. 8. The axi-symmetrical jet.

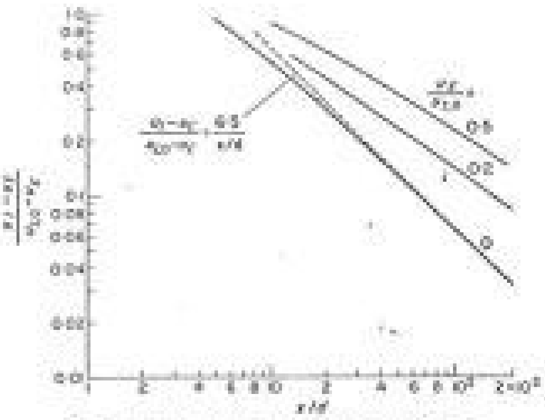

FIG. 9. Decay of centre-line velocity of the jet.

so that the extra amount of fluid entrained exact solution for the Prandtl 1925 mixing-length hypothesis. As a mathematical test of our procedure, we should expect good agreement between the present solution and Tollmien's solution. In Fig. 10 is presented the comparison of the dimensionless velocity profiles. We can conclude that use of only eleven grid lines has predicted very satisfactory velocity profiles.

4.3. *Radial wall jet*

Statement of the problem. As a final illustration, we present the results of a calculation of a radial wall jet. Though it is within the scope of boundary-layer theory, this case has several unusual features: the flow direction is at right angles to the axis of symmetry (i.e. $\beta = 90°$); the flow contains characteristics of both the conventional boundary layer and the free jet;

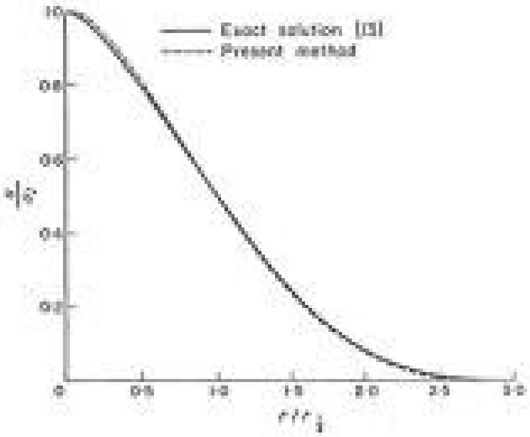

FIG. 10. Dimensionless velocity profile in the jet: comparison of the present and exact solutions.

the velocity profiles exhibit a maximum, and consequently the shear stress changes sign. In practice, such a flow occurs when a jet impinges normally on a plate. The problem here is to predict the development of the velocity profile for such a wall jet on smooth wall, starting from a known velocity profile at a given distance from the axis of symmetry.

Details of solution procedure. Once again the partial differential equation solved was (2.1.4). The effective viscosity was calculated by using Prandtl's 1925 mixing-length hypothesis. The variation of mixing length l_m was taken as:

$$\left.\begin{array}{ll} 0 < y \leqslant \dfrac{\lambda_G y_I}{\kappa}; & l_m = \kappa y; \\[2ex] \dfrac{\lambda_G y_I}{\kappa} \leqslant y: & l_m = \lambda_G y_I; \end{array}\right\} \quad (4.3.1)$$

where y_I is the characteristic thickness defined in Section 4.2 and κ and λ_G are constants. We have used: $\kappa = 0{\cdot}5$, and $\lambda_G = 0{\cdot}12$. These values will appear to be somewhat higher than those appropriate to conventional boundary layers or plane wall jets. However, experimental data for entrainment and shear stresses in *radial* wall jets do show that the corresponding mixing length must be larger.

For the first interval near the wall, we have assumed that the velocity profile corresponds to the "universal" law of the wall, given by:

$$u^+ = \frac{1}{\kappa}\ln(9y^+). \quad (4.3.2)$$

The shear stress at the wall can be calculated from this law, which incidentally is an example of the Couette-flow relationships mentioned in Section 2.3.

The number of grid lines used across the layer was sixteen and forward steps of one-fourth of the layer thickness were taken.

The calculations were performed for a particular set of experimental data taken from [14].

Results and comparison with experiment. Figure 11 shows our predictions and the experimental data for the decay of maximum velocity

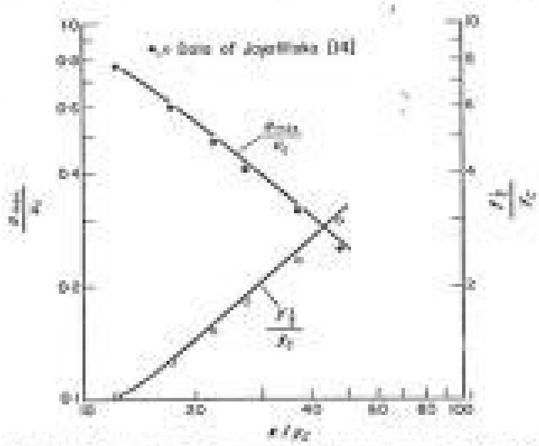

FIG. 11. Decay of maximum velocity and growth of half-value thickness of a wall jet.

and for the growth of the half-value thickness $y_{\frac{1}{2}}$. This half-value thickness is defined as the distance from the wall of a point which is beyond the maximum and at which the velocity is equal to one-half of the maximum velocity. The symbols y_C and u_C stand respectively for the thickness of the slot and the velocity at it in the experimental situation under consideration. The distance x is measured from the slot. The agreement with experiment is satisfactory in this case. Indeed the constants κ and λ_G have been chosen so as to obtain good agreement.

The integration takes about 0·2 min of IBM 7090 computer time.

5. CONCLUSIONS

1. The foregoing method of solving sets of simultaneous non-linear parabolic differential equations has proved itself to be convenient, accurate and quick in three rather different circumstances.
2. The main merits of the method derive from its use of the non-dimensional stream function as cross-stream variable and of a grid-control procedure ("entrainment law") which locally satisfies the differential equation of motion. Other features of the method, for example the linearisation of the finite-difference formulae, are inessential, and perhaps not particularly worthy of emulation.
3. Considerable simplification has been effected, for turbulent flows, by the neglect of longitudinal convection in the interval close to the wall; this permits the momentum and heat flux through the laminar sub-layer to be expressed by algebraic relations based upon once-for-all integrations or empirical laws.
4. Further development of the method should be directed towards the formulation of a general, optimum entrainment law, and the testing of the procedure for confined flows.

ACKNOWLEDGEMENTS

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REFERENCES

1. S. V. Patankar and D. B. Spalding, A calculation procedure for heat transfer by forced convection through two-dimensional uniform-property turbulent boundary layers on smooth impermeable walls, in *Proceedings of 3rd International Heat Transfer Conference*, Chicago 1966, Vol. II, p. 50.
2. L. Binder, *Doctoral dissertation of the Technische Hochschule, München.* Knapp, Halle (1911); and E. Schmidt, *Föppl's Festschrift*, p. 179. Springer, Berlin (1924).
3. J. Crank and P. Nicholson, A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type, *Proc. Camb. Phil. Soc. Math. Phys. Sci.* 43, 50 (1947).
4. V. M. Paskonov, A standard programme for the solution of boundary-layer problems, in *Numerical Methods in Gas Dynamics*, edited by G. E. Roslyakov and L. A. Chudov, pp. 74–79, NASA, Washington TTF-360, TT65-50138 (1966).
5. L. Prandtl, Bericht über Untersuchungen zur ausgebildeten Turbulenz, *Z. Angew. Math. Mech.* 5(2), 136–139 (1925).
6. A. N. Kolmogorov, Equations of the turbulent motion of an incompressible fluid, *Izv. Akad. Nauk SSSR Ser. Phys.* 6(1/2), 56–58 (1942).
7. L. Prandtl, Über ein neues Formelsystem für die ausgebildete Turbulenz, Nachrichten der Akad. Wiss. Gottingen, Math. Phys. 6 (1945).
8. A. S. Monin, Dynamic turbulence in the atmosphere, *Izv. Akad. Nauk.* 14(3), 232–254 (1950).
9. G. S. Glushko, Turbulent boundary layer on a flat plate in an incompressible fluid. *Izv. Akad. Nauk SSSR Mekh* (4), 13 (1965).
10. S. V. Patankar, Wall-shear-stress and heat-flux laws for the turbulent boundary layer with a pressure gradient, Imperial College, London, Department of Mechanical Engineering Tech. Note TWF/TN/14 (1966).
11. G. N. Abramovich, *The Theory of Turbulent Jets.* M.I.T. Press, Cambridge, Mass. (1963).
12. E. R. Van Driest, Investigation of laminar boundary layer in compressible fluids using the Crocco method, NACA TN 2597 (1952).
13. W. Tollmien, Berechnung turbulenter Ausbreitungsvorgänge, *Z. Angew. Math. Mech.* 6, 468–478 (1926); A' so translated as NACA TM 1085 (1945).
14. C. L. V. Jayatillaka, The influence of Prandtl number and surface roughness on the resistance of the laminar sublayer to momentum and heat transfer, Imperial College, London, Department of Mechanical Engineering, Report TWF/R/2 (1966).

Résumé—Un processus pas à pas, numérique, implicite et général est présenté pour la solution d'équations paraboliques aux dérivées partielles, et plus particulièrement de celles de la couche limite. La principale nouveauté réside dans le choix d'une grille qui ajuste son écartement de façon à s'adapter à l'épaisseur de la couche limite dans laquelle existent des gradients importants. La fonction de courant sans dimensions est employée comme variable indépendante dans la couche limite.

Les possibilités de la méthode sont montrées en l'appliquant à: la plaque plane chauffée dans un écoulement laminaire à nombre de Mach élevé; le jet turbulent à symétrie de révolution dans une atmosphère en mouvement ou au repos; et le jet pariétal turbulent radial.

Zusammenfassung—Es wird ein allgemeines, implizites, numerisches fortschreitendes Rechenverfahren angegeben, das zur Lösung partieller Differentialgleichungen vom parabolischen Typ, insbesondere der Grenzschichtdifferentialgleichungen, geeignet ist. Das wesentlich Neue an diesem Verfahren liegt in der Wahl eines Differenzengitters, das seine Schrittweite der Dicke der Schicht anpasst in welcher bedeutende Gradienten der Zustandsgrössen auftreten.

Die dimensionslose Stromfunktion dient als unabhängige Variable über die Grenzschicht.

Die Leistungsfähigkeit der Methode wird durch Anwendung auf folgende Probleme gezeigt: beheizte ebene Platte mit einer Laminarströmung hoher Mach-Zahl; axial symmetrischer turbulenter Freistrahl in bewegter und ruhender Umgebung; und radial turbulenter Wandstrahl.

Аннотация—Приводится численный метод решения дифференциальных уравнений параболического типа в частных производных применительно к задачам пограничного слоя. Новизна, в основном, относится к выбору сетки, ширина которой соответствует толщине слоя, в котором имеются значительные градиенты основных параметров.

Безразмерная функция тока используется в качестве независимой переменной поперек слоя. Возможности метода иллюстрируются следующими примерами: нагретая плоская пластина в ламинарном потоке при больших числах Маха; осесимметричная турбулентная струя в движущейся и неподвижной средах; радиальная турбулентная пристенная струя.

APPENDIX 'B'

Numerical Solution of the Elliptic Equations for Transport of Vorticity, Heat and Matter in Two-Dimensional Flow.

by

A K Runchal, D B Spalding and M Wolfshtein

High-Speed Computing in Fluid Dynamics
The Physics of Fluids Supplement II, pp 21-28, 1969.

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HIGH-SPEED COMPUTING IN FLUID DYNAMICS THE PHYSICS OF FLUIDS SUPPLEMENT II, 1969

Numerical Solution of the Elliptic Equations for Transport of Vorticity, Heat, and Matter in Two-Dimensional Flow

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Imperial College of Science and Technology, London, England

A finite-difference method is presented for the solution of the elliptic differential equations for the steady transport of momentum, heat, and matter in two-dimensional domains. Special features of the method include an unsymmetrical formulation for the convection terms, which promotes convergence at some cost in accuracy; obedience to the conservation equations for all subdomains; the use of Gauss-Seidel iteration procedure; employment of grids having nonuniform mesh; and a novel treatment of the boundary condition for vorticity. Solutions are presented for the laminar flow and heat transfer inside a square cavity with a moving top, an impinging jet, and a Couette flow with mass transfer. The influence of the Reynolds and Prandtl numbers, and of the impinging jet "free" boundary conditions is studied, and the results of the computations are shown to agree with existing physical knowledge. The influence of mesh size, mesh nonuniformity, and the vorticity wall boundary condition on convergence and accuracy is studied. It is shown that convergence may be secured for a wide range of Reynolds numbers with coarse-meshed grids. The convergence and computation speed appear to be satisfactory for many purposes; the accuracy of the solutions is discussed, and some improvements are suggested.

I. INTRODUCTION

This paper provides some new solutions of the elliptic equations which govern the distributions of velocity and temperature in steady, laminar, plane flows of a uniform-property fluid. Numerical solutions are obtained for three configurations: the square-shaped cavity with a sliding wall, a jet, impinging at right angles on an infinite wall, and a Couette flow with mass transfer. In these solutions, convergence is secured without any severe restriction on the mesh size of the finite-difference grid or the use of under/over-relaxation. We also wish to report some findings about the influences on accuracy and convergence of mesh size, grid-point distribution, and formulation of the vorticity boundary condition.

Many authors have concerned themselves with the finite-difference solution of the equations of steady-state motion and continuity in a two-dimensional domain. For example, Thom[1] obtained a solution for the flow around a circular cylinder in a uniform stream, and Allen and Southwell[2] extended this work. Other authors who have made notable contributions include Kawaguti,[3] Simuni,[4] and Burggraf.[5]

The last three authors found that their computational procedures failed to converge when Reynolds number became sufficiently large, unless the mesh size was steadily reduced or severe under-relaxation was used. In the present paper convergence is secured by the use of a new unsymmetrical finite-difference formulation. This formulation is unconditionally conservative, over any arbitrary control volume. Two other novelties are the use of nonuniform meshes, in order to improve the accuracy; and a new formulation for the wall vorticity evaluation, which is more accurate than earlier ones.

The results of the calculations are presented in two parts. Section III provides graphs which display the computed patterns of heat and fluid flow; fairly coarse grids have been used, so high accuracy is not claimed. Accuracy, and the influence on it of mesh size, of mesh-size distribution, and of the vorticity boundary conditions, are described in Sec. IV; problems of convergence and computer time are also discussed.

Of the three problems discussed, the square cavity is an important simplification of many recirculating flows. Together with the impinging jet, it has, therefore, much practical importance.

The third problem, that of Couette flow with mass transfer, is comparatively simple and is amenable to analytic techniques. Thus it is ideally suited for deducing conclusions about the accuracy and convergence of the method.

II. MATHEMATICAL FOUNDATIONS

A. Differential Equations and Boundary Conditions

We shall concern ourselves with the three equations which, for a uniform property, plane, steady flow, govern the distribution of stream function ψ, vorticity ω, and temperature T. These equations may

[1] A. Thom, Proc. Roy. Soc. London A141, 651 (1933).
[2] D. N. de G. Allen and R. V. Southwell, Quart. J. Mech. Appl. Math. 8, 129 (1955).
[3] M. Kawaguti, J. Phys. Soc. Japan 16, 2307 (1961).
[4] L. M. Simuni, Insh. Zh. 4, 446 (1964).
[5] O. R. Burggraf, J. Fluid Mech. 24, 113 (1966).

be written in vector form as

$$\operatorname{div}(\operatorname{grad}\psi) = -\omega, \tag{1}$$

$$V \times \operatorname{grad}\omega - \nu \operatorname{div}(\operatorname{grad}\omega) = 0, \tag{2}$$

$$V \times \operatorname{grad}T - \alpha \operatorname{div}(\operatorname{grad}T) = 0, \tag{3}$$

$$V = -i_z \times \operatorname{grad}\psi, \tag{4}$$

where V is the velocity vector, ν is the kinematic viscosity, α is the thermal diffusivity, and i_z is a unit vector normal to the plane.

The boundary conditions with which we shall be concerned are as follows: ψ is a known function along any wall, and Eq. (1) dictates that, along the wall

$$\frac{\partial^2\psi}{\partial n^2} = -\omega + (\text{a known function}), \tag{5}$$

where n is the normal distance from the wall. The temperature distribution along the walls will be taken as known.

The impinging-jet flow is supposed to occur in a semi-infinite medium. However, to simplify computation, "artificial" conditions will be invoked along the boundary of a finite domain of integration. These will be described in Sec. IIIB below.

B. Finite-Difference Equations and Boundary Conditions

We shall confine attention to the values of the variables which prevail at the nodes of a rectangular, nonuniform grid, covering the field of integration. Figure 1 displays a part of the grid and also, by dotted lines, the rectangular areas which surround each node. Attention is focused on a typical point P, and on the four surrounding points, N, S, E, and W.

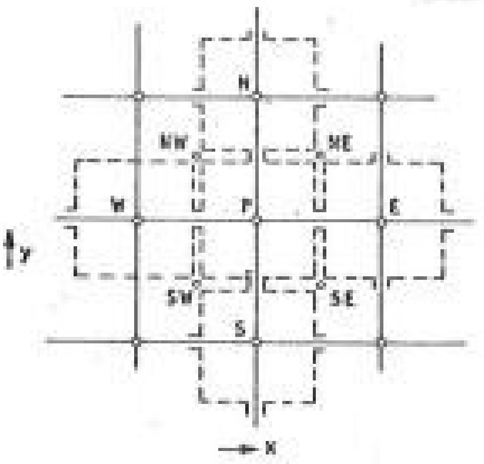

FIG. 1. Illustration of a part of the grid of points.

The finite-difference equations are derived from differential ones by integration over the rectangular areas shown in Fig. 1, together with assumptions about the distributions of the variables between grid points. The resulting finite-difference equations, corresponding, respectively, to (1), (2), and (3), are

$$\begin{aligned}(\psi_N - \psi_P)\frac{x_E - x_W}{2(y_N - y_P)} + (\psi_S - \psi_P)\frac{x_E - x_W}{2(y_P - y_S)} \\ + (\psi_E - \psi_P)\frac{y_N - y_S}{2(x_E - x_P)} \\ + (\psi_W - \psi_P)\frac{y_N - y_S}{2(x_P - x_W)} \\ = \frac{\omega_P(x_E - x_W)(y_N - y_S)}{4},\end{aligned} \tag{6}$$

$$\begin{aligned}(\omega_N - \omega_P)\left\{\frac{(\psi_{NE} - \psi_{NW}) + |\psi_{NE} - \psi_{NW}|}{2}\right\} \\ + (\omega_S - \omega_P)\left\{\frac{(\psi_{SW} - \psi_{SE}) + |\psi_{SW} - \psi_{SE}|}{2}\right\} \\ + (\omega_E - \omega_P)\left\{\frac{(\psi_{SE} - \psi_{NE}) + |\psi_{SE} - \psi_{NE}|}{2}\right\} \\ + (\omega_W - \omega_P)\left\{\frac{(\psi_{NW} - \psi_{SW}) + |\psi_{NW} - \psi_{SW}|}{2}\right\} \\ + \nu\left\{\frac{x_E - x_W}{2}\left(\frac{\omega_N - \omega_P}{y_N - y_P} + \frac{\omega_S - \omega_P}{y_P - y_S}\right)\right. \\ \left. + \frac{y_N - y_S}{2}\left(\frac{\omega_E - \omega_P}{x_E - x_P} + \frac{\omega_W - \omega_P}{x_P - x_W}\right)\right\} = 0,\end{aligned} \tag{7}$$

together with a further equation, such as Eq. (7) in form, but with T in place of ω and α in place ν.

The terms in the curly brackets of Eq. (7) deserve some comment. First, the subscripts, NE, NW, SE, and SW denote the four corners of the rectangle enclosing point P; the values of ψ at these points are to be taken as the arithmetic means of values at the four grid nodes lying nearest to them. This will insure that the resulting solutions satisfy the conservation equations over arbitrarily large or small portions of the domain of integration. Second, the contents of the curly brackets vanish when the ψ difference is negative; this means that there is a finite contribution of convection only for surfaces across which there is a positive rate of flow of fluid into the rectangle surrounding P. Consequently, the equation becomes positive definite, and convergence is secured.

The boundary regions are handled differently.

Grid nodes are set in the boundaries. If the boundary coincides with a solid wall, the ψ and T values of these are given, and the vorticity values are deduced from Eq. (5). We have tried two different ways to express this equation in finite-difference form.

(i) If we assume that the vorticity is constant near the wall, it follows that

$$\omega_P = -\frac{2(\psi_E - \psi_P)}{(\Delta y)^2} - \frac{2u_P}{\Delta Y}, \qquad (8)$$

where P is a boundary point, and E is a point adjacent to the boundary.

This formulation has been used by the majority of previous workers in the field; however, it usually implies an unrealistic discontinuity in ω at the point E.

(ii) If we assume that the vorticity varies linearly from P to E, it follows that

$$\omega_P = -\frac{\omega_E}{2} - \frac{3(\psi_E - \psi_P)}{(\Delta y)^2} - \frac{3u_P}{\Delta y}. \qquad (9)$$

Some other boundary conditions which have been used will be described in Sec. III below, in connection with the description of particular flows.

The above algebraic equations, taken together, define the mathematical problem. Their simultaneous solution is the task of the computational procedure which will now be outlined.

C. Outline of the Computational Procedure

Equation (7) can be rewritten as

$$\omega_P = A_N\omega_N + A_S\omega_S + A_W\omega_W + A_E W_E, \qquad (10)$$

where the coefficients A_N, A_S, etc., are given by

$$A_N = a_N/(a_N + a_S + a_E + a_W) \qquad (11)$$

and

$$a_N = \frac{(\psi_{NE} - \psi_{NW}) + |\psi_{NE} - \psi_{NW}|}{2} + \frac{\nu(x_E - x_W)}{2(y_N - y_P)}. \qquad (12)$$

For a discussion of the convergence of such sets of equations, the reader is referred to Barakat and Clark[6] and Runchal *et al.*[7]

Temperatures are now calculated from an equation similar in form to (10). Stream-function values are calculated from the equation which is obtained by rearranging Eq. (6) so as to only have ψ_P on the left-hand side. Values of vorticity at the solid boundaries are calculated from Eqs. (8) or (9).

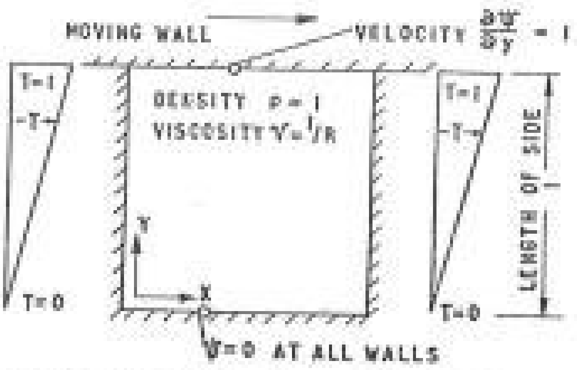

FIG. 2. Geometry and boundary condition of the square cavity problem.

III. THE COMPUTED FLOW PATTERNS

A. The Square Cavity with a Moving Wall

Figure 2 illustrates the problem to be solved. A fluid revolves steadily in a square-shaped cavity under the influence of the sliding upper wall. This wall is held at one temperature, the opposite at another; the side-wall temperatures vary linearly between those of the top and the bottom.

The computations were made with a rather coarse (13 × 13) but nonuniform grid. The grid used is indicated in just one of the diagrams in Fig. 3. For all the contours presented here, the boundary values of the vorticity were obtained from Eq. (9).

Figure 3 presents the results for Reynolds num-

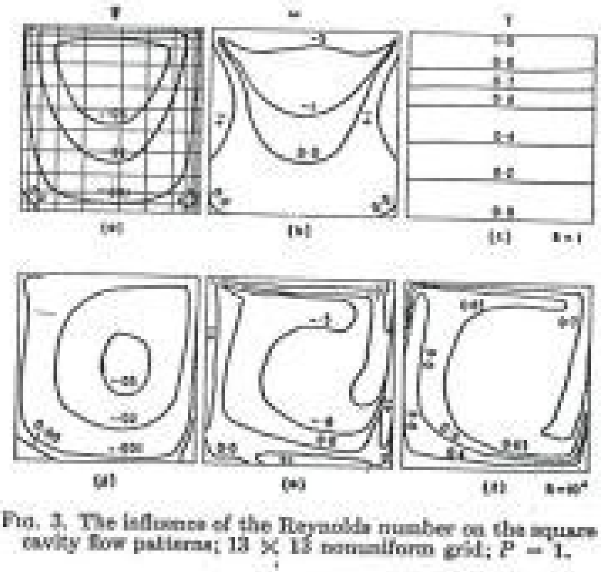

FIG. 3. The influence of the Reynolds number on the square cavity flow patterns; 13 × 13 nonuniform grid; $P = 1$.

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[6] H. Z. Barakat and J. A. Clark, in *Proceedings of the Third International Heat Transfer Conference* (American Institute of Chemical Engineers, New York, 1966), Vol. 2, p. 152.

[7] A. K. Runchal, D. B. Spalding, and M. Wolfshtein, Imperial College, Report No. SF/TN/14 (1968).

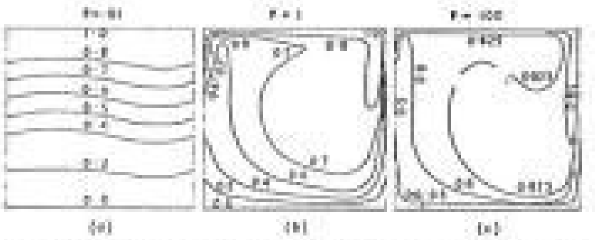


FIG. 4. The influence of the Prandtl number on the temperature contours; 13 × 13 nonuniform grid; $R = 1000$.

bers of 1 and 10^3, and a Prandtl number of unity. The contours reveal the existence of a large primary eddy in the cavity; this is cushioned by small contrarotating eddies in the two lower corners for all Reynolds numbers. The temperature distribution in the field at low R is almost the same as that in the walls; but at high R the temperature contours are caused to bulge and sag by the convective effect of the moving fluid. The vorticity contours are similarly distorted at high Reynolds numbers from the near-symmetrical form which they possess when R equals unity.

Figure 4 shows the temperature contours only, for a single Reynolds number 10^3 and three different Prandtl numbers 0.01, 1, and 100.

Evidently, the high thermal conductivity, which causes the low Prandtl number, nearly succeeds in preventing the convective processes from distorting the temperature contours from their linear, low Reynolds number form. On the other hand, when the thermal conductivity is low, as when P equals 100, the distortions are still more pronounced than for a Prandtl number of unity. These results are easy to understand.

The qualitative features of the above results present no surprises. They are in conformity with the earlier but less extensive predictions of Squire,[8] Batchelor,[9] Mills,[10] and Burggraf.[5] All the numerical solutions now available suggest that secondary eddies should appear in the lower corners of the cavity, even for creeping flows. In this connection it is of interest to note that Macagno and Hung[11] made the same observation for a captive annular eddy behind a downstream-facing step in a pipe.

B. The Plane Jet Impinging on a Wall

Figure 5 illustrates the geometry and boundary conditions of the problem. The symmetrical plane jet is supposed to discharge vertically downward on to a horizontal plate. Far away from the plate, the velocity profile inside the jet is known from free-jet theory; and, at the plate itself, both horizontal and vertical components of the velocity are equal to zero.

We have confined our attention to a square region with one side on the centerline of the jet, and of length equal to the free-jet width. We have employed two alternative procedures on the free boundary of the region: In case (a) we have required the stream lines to cross the free sides of the control volume at right angles, and the fluid entering the control volume from these boundaries to have zero vorticity; and in case (b) the free boundary was taken as closed by a wall. In both cases we have taken the free-jet temperature to be equal to the ambient, and different from the plate temperature; we also have required that there should be no diffusion of heat and vorticity across the vertical walls of the control volume.

For all computations, a 13 × 13 nonuniform mesh was used; the distribution is indicated in Fig. 6. The vorticity at the solid wall was obtained from Eq. (9).

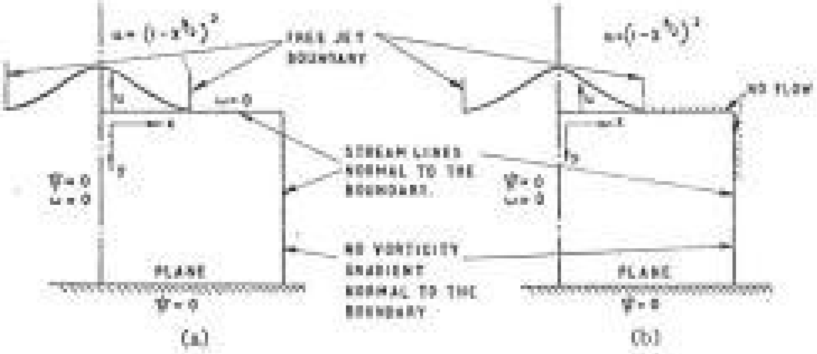


FIG. 5. Illustration of the impinging jet problem with two alternative sets of boundary conditions at the "free" boundary.

[8] H. B. Squire, J. Roy. Aeron. Soc. 60, 203 (1956).
[9] G. K. Batchelor, J. Fluid Mech. 1, 714 (1956).
[10] R. D. Mills, J. Roy. Aeron. Soc. 69, 714 (1965).
[11] E. O. Macagno and T. K. Hung, J. Fluid Mech. 28, 43 (1967).

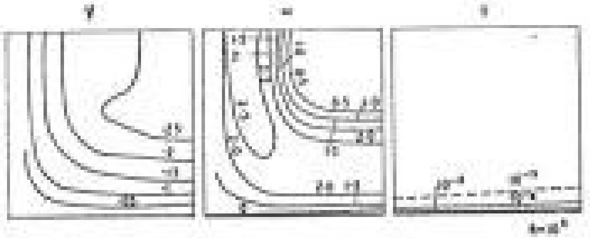
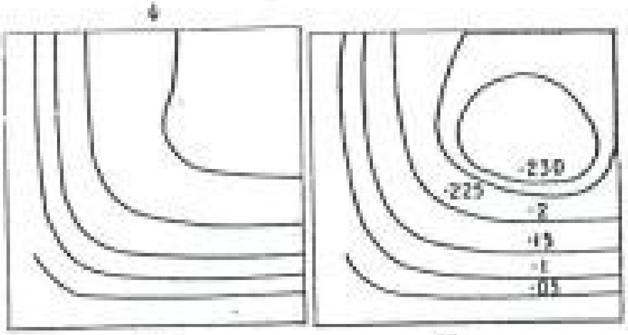


FIG. 6. The impinging jet flow patterns; 13 × 13 nonuniform grid; $R = 10^3$, $P = 1$.



(a) (b)

FIG. 7. The influence of the "free" boundary conditions on the impinging jet flow patterns; 13 × 13 nonuniform grid; $R = 1000$.

The Reynolds number of the flow is defined by reference to the maximum velocity and to the width of the free jet where it enters the control volume. The results of the computations at $R = 10^3$ and, for Prandtl number of unity, are displayed in Fig. 6. Once again ψ, ω, and T contours are the means of display. The results display the qualitative features that experimental observations lead one to expect. The fluid from the jet is deflected by the plate and flows along it. Fluid from the surrounding atmosphere is set in motion by the shear stress at the jet boundary; it is then drawn toward the jet, and carried along with it. The major part of the temperature variation is confined to a thin region close to the wall; this boundary layer tends to thicken in the downstream direction.

Figure 7 shows the ψ pattern for a Reynolds number of 10^3, with the two types of boundary conditions. It is obvious that the differences in the main fast-moving part of the jet are very small. The results confirm the qualitative expectation which physical experience suggests: the obstruction or constraint of the slow-moving fluid, which is entrained from the surroundings, has very little effect when the obstruction is not too severe.

IV. SOME MATHEMATICAL ASPECTS OF THE SOLUTIONS

A. Accuracy

1. *False Diffusion*

For a Reynolds number of 10^3, the exact solutions of the equations must show the vorticity of the fluid to be constant along every streamline. However, Fig. 6 shows that our numerical solutions do not fulfil this expectation; it appears that the numerical procedure has the effect of introducing an additional "false" diffusion of vorticity, which, at high Reynolds numbers, may be greater in magnitude than that brought about by the true viscosity.

Detailed considerations of this effect, some of which were described by Wolfshtein,[12] have led us to the following conclusions:

(i) All one-sided finite difference schemes, which are known to us, suffer from this defect.

(ii) The magnitude of this "false" diffusion effect for a square of mesh of size h, and a uniform velocity u can be expressed by $\nu_{false} \simeq 0.36\, uh \sin(2\theta)$, where θ is the angle that the streamlines make with the coordinate system, and ν_{false} is the kinematic viscosity which is responsible for this false diffusion of vorticity.

Obviously, the error caused by the false-diffusion process may be appreciable, especially when the Reynolds number is high, the mesh size is large, and θ is near 45°; however, this error need not be regarded as unacceptably large. Figure 8 displays the velocity profile across the vertical central plane of the square cavity, at $R = 10^2$, deduced both from our work and from that of Burggraf and Mills. The curve ascribed to Burggraf can be regarded as nearest to the exact solution, since it was computed with a 51 × 51 grid; yet our solution, obtained with a 13 × 13 nonuniform grid, and Mills' solution obtained with a 15 × 15 uniform one, differ very little. To some extent, it reflects the fact that, except near the corners, the streamlines in the square cavity run almost parallel to the mesh. Secondly, where the streamlines are appreciably inclined to the mesh, the velocities are fairly small.

2. *The Influence of Mesh Size and Distribution*

The mesh-size effect is shown in Fig. 9, which shows the velocity gradient at the wall beneath the impinging jet against distance from the axis. The curves marked I, II, and III show how, for a uniform mesh, the velocity-gradient profile is influenced

[12] M. Wolfshtein, Ph. D. thesis, London University (1968).

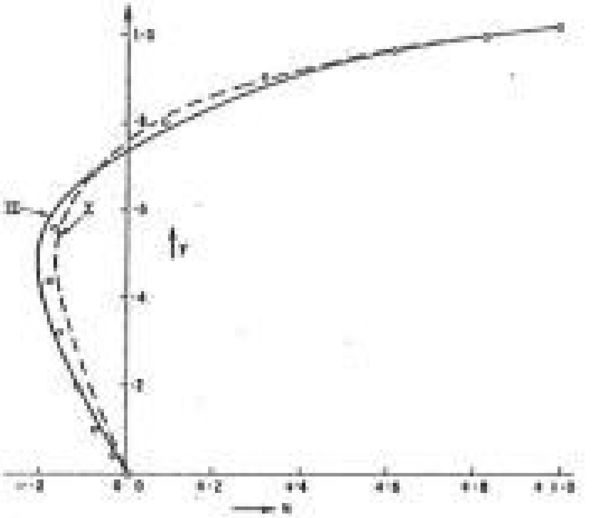


FIG. 8. Comparison of some available results of the velocity profile across the vertical central plane of the square cavity; $R = 100$.

Symbol	Grid	Source
I — — —	15 × 15 Uniform	Mills
II ———	Up to 51 × 51 Uniform	Burggraf
0	13 × 13 Nonuniform	Present

by size. The boundary condition employed was Eq. (8) rather than (9), so the solution for the 11 × 11 mesh is excessively inaccurate, but the influence of mesh size on accuracy appears, therefore, all the more clearly.

Figure 9 enables one to see that, if the grid is arranged so that the points are closest together near the wall where the vorticity and temperature vary most rapidly, then the accuracy is appreciably higher than when the mesh size is uniform. This conclusion follows from the nearness of curve IV, which was obtained with an 11 × 11 nonuniform mesh, to curve III, the most accurate of those obtained with a uniform grid, and to curve VI, which was obtained with a 21 × 21 nonuniform grid. While we are not yet in a position to propose general rules for mesh distribution, it can certainly be concluded that substantial economies in computer time can be effected.

3. The Influence of the Boundary Conditions

A further lesson can be learned from inspection of Fig. 9. Curve V represents the wall-velocity-gradient distribution calculated with an 11 × 11 *uniform* mesh, and with boundary condition (9) in place of (8). While the curve is still much lower than that with 41 × 41 mesh, the errors are appreciably less than those associated with the boundary condition of curve I. *It is* Eq. (9), it will be remembered, which has been employed in the computations displayed

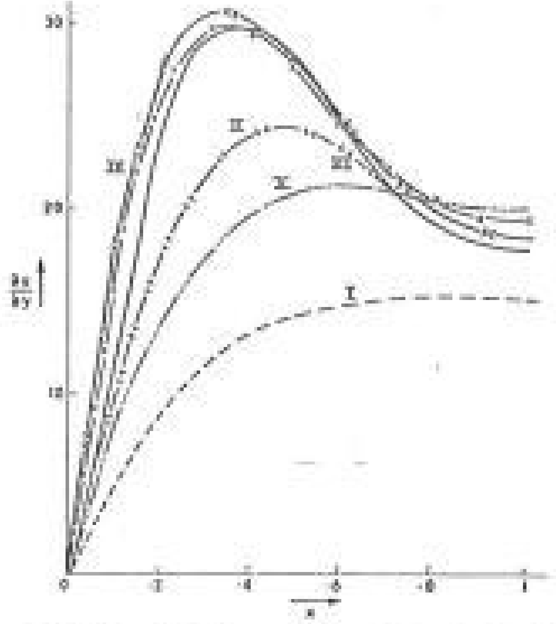


FIG. 9. Variation with distance of the velocity gradient along the wall of the impinging jet; $R = 1000$.

Symbol		Grid	Hypothesis for vorticity boundary conditions
I	— — —	11 × 11 Uniform	Uniform vorticity
II	—++—	21 × 21 Uniform	Uniform vorticity
III	———	41 × 41 Uniform	Uniform vorticity
IV	—×—	11 × 11 Nonuniform	Uniform vorticity
V	—·—·—	11 × 11 Uniform	Linear vorticity
VI	—××—	21 × 21 Nonuniform	Uniform vorticity

in Sec. III; so, since in addition the grids were nonuniform, those solutions can be taken as being fairly accurate. Figure 8 is, indeed, a confirmation of this.

4. Couette Flow with Mass Transfer

We can throw more light on the problem of accuracy by presenting the results of some computations, performed using the general computer program for a one-dimensional situation; the flow between two parallel porous plates, one of which moves in its own plane, when the pressure gradient is zero. The rates of flow out of one wall and into the other, are equal and given the symbol M; the distances between the plates, the relative velocity of the plates, the fluid density, and the fluid viscosity will all be taken as unity. This situation allows an exact solution, namely,

$$u = -\exp(M)[\exp(Mz - 1)]/\exp(M - 1). \quad (13)$$

Figure 10 shows how the accuracy of the solution is influenced by the number of intervals N into which the interplate distance is uniformly divided, for various values of the flow rate M, and for each of the two boundary conditions. The accuracy is expressed through the ratio of the computed value of the vorticity at the $x = 1$ wall to the exact value of this vorticity; this ratio should of course equal unity.

Inspection of Fig. 10 teaches the same lessons which we have already learned: accuracy increases with increase in N, and Eq. (9) is a better boundary condition than Eq. (8). We also see that the accuracy deteriorates as M increases, presumably because, when M is large, very steep vorticity gradients appear near the $x = 1$ wall. We have tried to solve this problem with nonuniform meshes and found that the accuracy was improved by specifying finer meshes near the upper wall. However, even a very nonuniform mesh could not remove all the inaccuracies.

Further investigation revealed that these inaccuracies may be attributed to the use of linear distributions to represent curves which are in fact exponential. The following two steps were found to give a much improved accuracy in the Couette flow problem. First, we require that the grid point should always be pressed against the wall through which the fluid is leaving the cell. Second, we evaluate the diffusion through the cell wall by fitting an exponential profile to the vorticity or temperature. By this method accuracy was improved, without losing the convergence. For a more detailed discussion the reader is referred to Runchal *et al.*[7]

In two-dimensional situations, we believe that similar procedures may be devised. However, we have not yet completed this test, and we shall, therefore, leave its discussion to future papers.

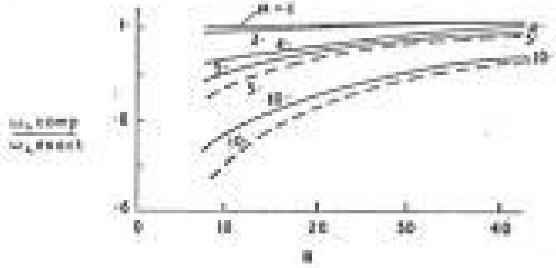


Fig. 10. The influence of N, M, and the vorticity boundary condition on the accuracy of the finite-difference computations of the Couette flow; uniform mesh. The solid lines represent Eq. (9). The dashed lines represent Eq. (8).

B. Convergence

The prime advantage claimed for the present formulation of the finite-difference equations is that it procures convergence of the successive-substitution procedure. Its superiority in this respect is sufficiently demonstrated by the fact that the comparison in Fig. 11 is for $R = 100$, rather than the $R = 10^4$ adopted in Fig. 3 since divergence has prevented earlier authors from obtaining solutions even at a Reynolds number of 1000.

We found it better to increase the interval a little at each row than to have a block of small intervals immediately adjacent to another block of intervals of double the size. There is reason to believe that the size ratio of neighboring intervals should not exceed about 1.5.

Although Eq. (9) almost always gives better accuracy than Eq. (8), it is also more liable to provoke divergence, especially when the grid intervals are unequal, and when the Reynolds number is high. The cure for this divergence is to remove the wall vorticity from the array of successively substituted variables, and to take account of Eq. (9) in the substitution formula for the vorticities at points one interval away from the wall.

We assumed that convergence was obtained when the change in the value of any variable between successive iterations was less than 0.0001 of the maximum value of this variable in the whole field. The number of iterations required increased rapidly with the number of grid nodes under consideration. We also found that the computing times for the square cavity are considerably longer than those for the impinging jet, especially at high Reynolds number. We believe that this difference is a result of the fact that in closed streamline flow errors may be carried out of the control volume by diffusion only, and this becomes increasingly ineffective as the Reynolds number is increased. Therefore, in such a case these errors are likely to be recirculated inside the field for a long time and thus prevent, or at least slow down, the convergence.

V. CONCLUSIONS

(a) The present finite-difference scheme provides equations which are soluble by successive substitution over a wider range of conditions than hitherto possible. An unsymmetrical treatment of the convection term insures that only upstream values of the variables affect the values to be substituted at any grid point.

(b) Equation (9) gives more accurate results than Eq. (8), as a rule. It is also more liable to provoke divergence, but this can be countered by algebraic elimination of the wall vorticity from the substitution formulae.

(c) When steep gradients of fluid variables are a feature of the solution, the best accuracy is obtained when the grid points are distributed so as to be close together in the steep-gradient region.

(d) Even when all the recommendations are followed simultaneously, the solutions that are obtainable at a modest cost in computer time are not always sufficiently accurate for practical purposes. In future work it seems desirable to take account of the exponential nature of the solution.

ACKNOWLEDGMENT

The financial support of Imperial Chemical Industries (India) Pvt. Ltd. is gratefully acknowledged by A. K. Runchal.

APPENDIX 'C'

A Calculation Procedure for Heat, Mass and Momentum Transfer in Three-Dimensional Parabolic Flows.

by

S V Patankar and D B Spalding

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A CALCULATION PROCEDURE FOR HEAT, MASS AND MOMENTUM TRANSFER IN THREE-DIMENSIONAL PARABOLIC FLOWS

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Abstract—A general, numerical, marching procedure is presented for the calculation of the transport processes in three-dimensional flows characterised by the presence of one coordinate in which physical influences are exerted in only one direction. Such flows give rise to parabolic differential equations and so can be called three-dimensional parabolic flows. The procedure can be regarded as a boundary-layer method, provided it is recognised that, unlike earlier published methods with this name, it takes full account of the cross-stream diffusion of momentum, etc., and of the pressure variation in the cross-stream plane. The pressure field is determined by: first calculating an intermediate velocity field based on an estimated pressure field; and then obtaining appropriate correction so as to satisfy the continuity equation. To illustrate the procedure, calculations are presented for the developing laminar flow and heat transfer in a square duct with a laterally-moving wall.

NOMENCLATURE

- A, B, C, coefficients in the finite-different equations;
- d, dimension of the duct cross-section (Fig. 6);
- D, coefficient of the pressure-gradient term;
- F, a body force, equation (2.2) etc.;
- F_U, forward flow at upstream station, equation (3.3);
- F_D, forward flow at downstream station, equation (3.3);
- J, diffusion flux, equation (2.5);
- L^y, L^z, lateral flows defined by equation (3.3);
- m_p, a mass source defined by equation (2.13);
- $\dot{m}$, mass-flow rate through the duct;
- p, pressure in the cross-stream momentum equations;
- $\bar{p}$, pressure in the main-direction momentum equation;
- Pr, the Prandtl number;
- Re, a Reynolds number based on the duct side d;
- S, the source term in equation (2.5);
- S_U, S_P, finite-difference expressions representing the source term, equation (3.3);
- t, temperature;
- $\bar{t}$, the bulk temperature;
- T^y, T^z, transport coefficients defined by equation (3.3);
- $\tilde{T}$, a modified form of T defined by equation (3.6);
- u, velocity component in the x direction;
- v, velocity component in the y direction;
- w, velocity component in the z direction;
- x, distance in the main-flow direction;
- y, z, the cross-stream co-ordinates;
- δy, δz, distances between neighbouring grid points (Fig. 5);
- Δx, size of the forward step (Fig. 4);
- Δy, Δz, cross-stream dimensions of the control volume (Fig. 5);
- Γ, transport property in equation (3.1);

ρ, density;
τ, shear stress;
ϕ, a general dependent variable.

Superscripts
p, pressure;
u, v, w, corresponding velocity components;
ϕ, the dependent variable ϕ;
$*$, an estimated pressure;
$'$, the correction;
I, first phase of the double sweep;
II, second phase of the double sweep.

Subscripts
D, downstream station;
e, the point e in Fig. 5;
E, the point E in Fig. 3 or 5;
in, inlet to the duct;
max, maximum over the duct cross-section;
n, N, P, s, S, the corresponding points in Figs. 3 or 5;
u, the velocity u;
U, upstream station;
v, the velocity v;
w, the velocity w, or the point w in Fig. 5;
W, the point W in Fig. 3 or 5;
wall, the moving wall;
xy, xz, the corresponding co-ordinate planes;
ϕ, the dependent variable ϕ.

1. INTRODUCTION

1.1 *The purpose of the present paper*

Boundary-layer theory is one of the most advanced and popular of all the branches of fluid mechanics. Text-books describe it; research workers add daily to its repertoire of methods and store of experimental knowledge; and students and their teachers find it an unfailing source of educational exercises and of subjects for minor publications. Yet, from the point of view of engineering practice, the fruits of boundary-layer theory must be judged disappointing; despite the decades of development, the flow in an engine intake or over an aircraft fuselage, for example, must be determined, if at all, from experiment rather than calculation.

The reason is that almost all practically important boundary layers are three-dimensional. Even in the laboratory, the efforts of skilled experimenters fail to achieve sufficient two-dimensionality to allow adequate comparison with two-dimensional prediction procedures. (Extensive evidence of this is to be found, for example, in the proceedings of the 1968 Stanford Conference [1].) General techniques that are currently available for predicting boundary-layer phenomena on the other hand are exclusively two-dimensional in character.

It is true that a few techniques exist which may be applied to some special three-dimensional flows, and that these may have a limited success in predicting phenomena of the relevant class. However, the engineer needs general and flexible techniques, to which arbitrary initial and boundary conditions can be supplied in a straightforward way; and which yield predictions of velocities, of temperatures, of concentrations, and of the corresponding fluxes, without fuss or the supply of special insight.

It is in the nature of the problem that such techniques can be only of the finite-difference variety. Such numerical techniques exist for two-dimensional flows (e.g. Patankar and Spalding [2]); it is the purpose of the present paper to report the development of, and to describe and illustrate, a numerical procedure for predicting boundary-layer phenomena which are three-dimensional.

1.2 *Statement of the problem*

Definition. Here it must be made clear that we use the term "boundary layer" in a more general sense than is usual in the literature. We apply the term to all the flows which can be adequately described by differential equations that are parabolic in one distance co-ordinate. Thus, we call a flow a boundary layer,

(a) if there exists a predominant direction of flow (i.e. there is no reverse flow in that direction),

(b) if the diffusion of momentum, heat, mass, etc. is negligible in that direction, and

(c) if the downstream pressure field has little influence on the upstream flow conditions. When these conditions are satisfied, the co-ordinate in the main flow direction becomes a "one-way" co-ordinate; i.e. the upstream conditions can determine the downstream flow properties, but not vice versa. It is this convenient behaviour of the boundary-layer flows that enables us to employ a marching integration from an upstream station to a downstream one.

Some readers may feel that this extension of the term "boundary layer" is inconvenient or unwarranted. It is for this reason that we use, in the title of the paper, the more precise but unfamiliar term "parabolic flows".

Example. In order to appreciate the main features of these flows, it is useful to consider the situation illustrated in Fig. 1. Air flows steadily through a duct of rectangular cross-section; through the floor of the duct there penetrates a jet of a different fluid, say steam, which is blown obliquely along the wall. Downstream of the injection plane, the steam mixes with the air; and the interchanges of momentum between the two streams co-operate with the pressure gradient along the duct and the friction on the walls to produce in the mixture a swirling motion which decreases in intensity with longitudinal distance. The task of our three-dimensional boundary-layer theory is to predict this process, and all that is connected with it.

Figure 2 clarifies the matter further, by exemplifying some of the quantities which the prediction procedure must supply. Figure 2a shows how the mixing of the steam and air produces variations of the steam concentration that would be detected by analysis of the mixture clinging to the floor of the duct. This lateral

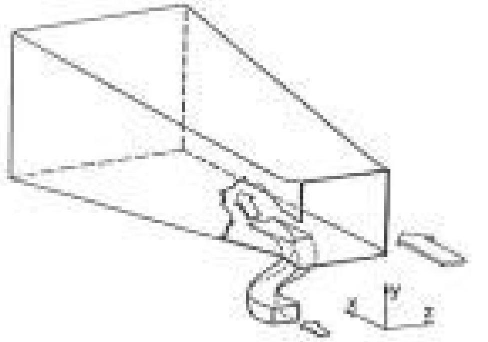

FIG. 1. Illustration of a three-dimensional parabolic flow.

spread of steam is the result of both convection and diffusion in the z direction; so we must be sure that both these processes are represented in the equations which are solved.

Figure 2b sketches the variation with longitudinal distance x of the space-average pressure across the duct, $\bar{p}$. This quantity is indicated as rising at first, in response to the injector-like action of the jet; thereafter it falls, as a consequence of friction on the walls. The sketch reminds us that $\bar{p}$ must be calculated; we do not know, as we do in some external boundary-layer situations, the pressure variation before the start of the computation.

Also to be computed is the variation of the longitudinal velocity component, u. Figure 2c illustrates, by a contour diagram, the probable form of this variation at the outlet section; the highest value of u appears in the bottom right-hand corner, because of the oblique injection of the fast-moving steam jet. The u variation is influenced by the gradient of the longitudinal momentum, by the shear stresses on the xy and xz planes, and by the convection of momentum from upstream. The differential equation governing u must express these influences individually and in simultaneous action.

Finally, Fig. 2d represents, by way of a set of vectors, the motion of the fluid in the plane of the duct outlet; it shows a large general vortex, in the sense resulting from the oblique injection of steam, with minor vortices of opposite sense in two of the corners. The values of the velocity components, v and w, are the result of the interaction with the convected moments of the shear stresses and the normal stresses on the xy and xz planes. It is therefore necessary to take these stresses into account in the computation; and the calculations of the pressure gradients will, it is clear, play a crucial part in the procedure.

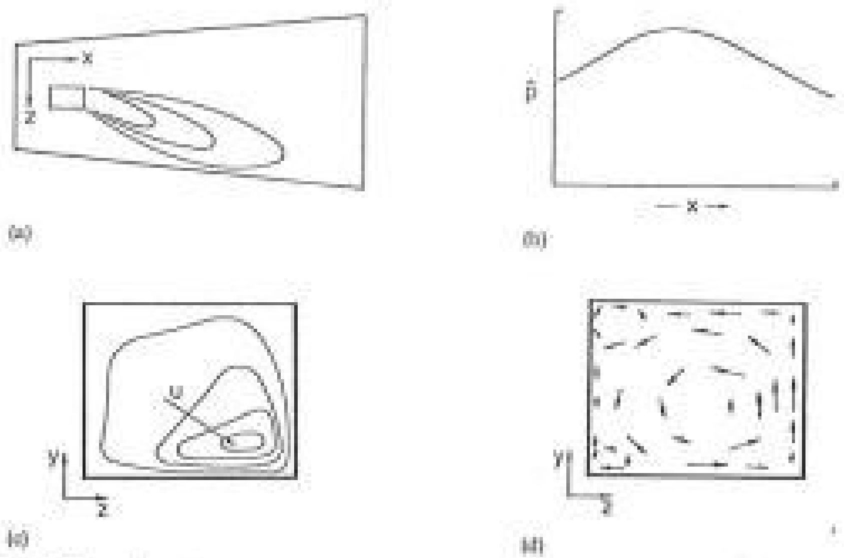

FIG. 2. Illustration of some of the quantities to be predicted by the calculation procedure. (a) Contours of steam concentration at the floor of the duct. (b) Variation of mean pressure with longitudinal distance. (c) Contours of longitudinal velocity at the outlet plane. (d) Velocity vectors in the outlet plane.

1.3 *Some remarks about previous work*

There are a few papers in the literature which report finite-difference procedures for solving the three-dimensional boundary-layer equations. The first appears to be that of Raetz [3]; but no use of the method has been reported. In recent years, Hall [4], Dwyer [5], Fannelop [6], Nash [7], Krause *et al.* [8], and Wang [9] have developed calculation procedures for the three-dimensional boundary layer outside of a solid body.

Although satisfactory, no doubt, for the particular purposes which their authors had in mind, none of these methods will solve our general problem. The reasons are: firstly, the procedures neglect the stresses and diffusion fluxes across either the xy or xz plane; and secondly, they do not take full account of the pressure variations in the yz plane. These omissions rob the model of precisely those agents which, in many circumstances, have the most significant effect. Moreover, since all these procedures have been applied to only *external* boundary layers, they do not provide any means of calculating the unknown pressure gradient in a confined flow.

Miller [10] has described a procedure which would indeed solve our general problem; he has applied it to the developing flow in ducts of arbitrary cross-section. His procedure, however, does not take advantage of the boundary-layer character of the flow, but treats the equations as elliptic in all the three space co-ordinates. Thus, Miller needs three-dimensional computer storage, the downstream boundary conditions, and excessive computer time. While looking for a method for *boundary-layer flows*, we should regard Miller's method as unnecessarily complex and hence unsuitable for our purposes.

When the available procedures in the special field of our enquiry are so seriously restricted (or complicated), it is helpful to look for guidance in related fields. Specifically, since steady three-dimensional flows and unsteady two-dimensional ones have several mathematical features in common, it is useful to enquire as to what methods have been employed for the latter brand of parabolic differential equations. There is a large literature on this subject, usefully digested by Harlow [11]. The papers most relevant to our present subject are those of Harlow and Welch [12], Amsden and Harlow [13], and Chorin [14]. These authors all use finite-difference procedures in which the dependent variables are the velocity components and the pressure (or a closely related quantity); the pressure is deduced from an equation which is obtained by the combination of the continuity equation and the momentum equations; and the idea is present of a first approximation to the solution, followed by a succeeding correction. It will later be seen that the method of the present paper shares these features.

It is appropriate to mention also some earlier work by the authors and their colleagues. Their two-dimensional boundary-layer procedure [2], when used for flows confined in ducts, involved calculating the pressure from the continuity equation by a non-iterative self-correcting process. This feature, not wholly unlike that of methods in the previous paragraph, will be employed below. Secondly, two procedures for three-dimensional boundary layers have just recently been developed (Caretto, Curr and Spalding [15]); one of these solves the same equations as the present method, *albeit* in a different manner; the other suppresses the pressure as a main variable, in favour of the x-direction vorticity. The present method is a rival to these two recent methods, and, it now appears, a successful one.

1.4 *Outline of the present paper*

The description of a numerical procedure for solving simultaneous equations can have two distinct aims, which it is seldom possible to accomplish simultaneously. The first aim is to convey to the reader the main principles, and the crucial tricks, and to leave him with the feeling that he could work out the rest for himself; the second is to present the particular equations, and to list the steps needed for their solution, with sufficient precision of detail to enable a computer programmer to begin his work.

Because the latter aim requires the equations to be written out in full, and because this entails a proliferation of subscripts that impede smooth reading and inhibit understanding, its fulfilment is deferred to a later section (Section 3); and even there the treatment is curtailed.

In Section 2 however, an attempt will be made to fulfil the first aim. Just sufficient of the details will be presented to convey the essential ideas; and the inessential features will be suppressed.

The second aim is difficult to fulfil within the normal length of a paper. Advantage is, therefore, taken of the fact that the procedure to be described here has many details in common with the present authors' two-dimensional procedure, which has been more completely reported [2]. Thus, the details given in Section 3 are by way of examples, and should be generalized and completed by reference to [2].

Section 4 describes an application of the procedure to the flow in a square duct with a laterally-moving wall.

2. MAIN FEATURES OF THE CALCULATION PROCEDURE

2.1 *The differential equations*

The equations. We can now express the problem described in Section 1.2 as that of solving the following equations, written with reference to the Cartesian co-ordinates x, y, z:

Continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (2.1)$$

Momentum:

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu) = \frac{\partial \tau_{u,xz}}{\partial y} + \frac{\partial \tau_{u,xy}}{\partial z} - \frac{\partial \bar{p}}{\partial x} + F_u \qquad (2.2)$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2) + \frac{\partial}{\partial z}(\rho wv) = \frac{\partial \tau_{v,xz}}{\partial y} + \frac{\partial \tau_{v,xy}}{\partial z} - \frac{\partial p}{\partial y} + F_v \qquad (2.3)$$

$$\frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho w^2) = \frac{\partial \tau_{w,xz}}{\partial y} + \frac{\partial \tau_{w,xy}}{\partial z} - \frac{\partial p}{\partial z} + F_w \qquad (2.4)$$

Other conservation equations (general form):

$$\frac{\partial}{\partial x}(\rho u\phi) + \frac{\partial}{\partial y}(\rho v\phi) + \frac{\partial}{\partial z}(\rho w\phi) = -\frac{\partial}{\partial y}(J_{\phi,xz}) - \frac{\partial}{\partial z}(J_{\phi,xy}) + S_\phi \qquad (2.5)$$

In regard to these equations, it is necessary to explain both what is included and what is omitted. As to symbols, ρ stands for density, τ for shear stress, J for diffusion flux, F for a body force; the symbol ϕ can stand for any property which can be convected and diffused, for example, stagnation enthalpy, chemical-species concentration, and turbulence energy; S_ϕ is the corresponding volumetric source rate. The subscripts u, v and w indicate which component of the momentum is in question; the subscripts xy and xz denote the planes on which the stresses or fluxes act.

The omissions from the equations are the shear stresses and diffusion fluxes acting on the yz plane. These omissions accord with our definition of a boundary layer and with the consequent necessity to ensure that no influence from downstream can penetrate upstream; stresses and fluxes on the yz plane would allow such an influence.

The uncoupling of longitudinal and lateral pressure gradients. A further point to note is that the symbol $\bar{p}$ used for the pressure in the x-momentum equation (2.2) is different from the symbol p in the two other momentum equations. This is a reminder of the fact that in our calculation procedure an inconsistency is deliberately introduced into the treatment of pressure, and that the quantities $\bar{p}$ and p are calculated differently. The pressure $\bar{p}$ can be thought of as a form of space-averaged pressure over a cross-section, and the gradient $\partial\bar{p}/\partial x$ is supposed to be known (or calculated) before we proceed to get the lateral pressure gradients $\partial p/\partial y$ and $\partial p/\partial z$. (The reader may find this point difficult to understand and appreciate at first; it should become clearer after perusal of Section 2.4 below.)

This practice is implicit in two-dimensional boundary-layer theories also; but it escapes notice because there is no necessity to solve the momentum equation for the cross-stream direction. Here we have two cross-stream directions; and we must solve the momentum equations for both of them, in order to find out how the fluid distributes itself between these two directions.

The practice is a necessary consequence of our intention to exploit the boundary-layer nature of the flow; it is the final step to be made in preventing downstream influences from propagating upstream. If the step is omitted, the result is not increased in accuracy, as one might naively expect; it is often a solution which is wholly unrealistic physically. The inconsistency in the treatment of pressure, it may be said, is one part of the price we pay for making the equations parabolic; the gain is the freedom to employ marching integration, and to use two-dimensional computer storage, even though the flow is three-dimensional and the full equations are elliptic.

Auxiliary information. The differential equations do not alone specify the problem; we need additional information of two kinds: initial and boundary conditions for all the dependent variables (u, v, w, p, ϕ); and auxiliary equations allowing the density, sheer stresses, diffusion fluxes, body forces and sources to be computed in terms of the dependent variables at each point in the field. Since this information is of the same kind as is needed for two-dimensional boundary layers, we shall treat it as well known, and allow it to be exemplified without preface in the subsequent discussion.

2.2 *The finite-difference equations*

The "staggered grid". Figure 3 shows how the points are arrayed in the yz plane at which are stored the variables u, v, w, p and ϕ. The boomerang-shaped envelopes enclose the triads of points denoted by a single letter, N, S, E, W, or P. This arrangement, which is similar to the one used by Harlow *et al.*, has the convenient feature

FIG. 3. The staggered grid.

that the cross-stream velocities v and w are stored at just the points at which they are needed for the calculation of the convective contribution to the balances of u and ϕ; and the pressures are stored so as to make it easy to calculate the pressure gradients which affect v and w.

The finite-difference equations. The differential equations of Section 2.1 can be expressed in the following finite-difference form:

$$C^v\{(\rho v)_N - (\rho v)_P\} + C^w\{(\rho w)_E - (\rho w)_P\} = C^u\{(\rho u)_{P,U} - (\rho u)_{P,D}\}. \quad (2.6)$$

$$u_P = A^u_N u_N + A^u_S u_S + A^u_E u_E + A^u_W u_W + B^u + D^u(\partial\bar{p}/\partial x). \quad (2.7)$$

$$v_P = A^v_N v_N + A^v_S v_S + A^v_E v_E + A^v_W v_W + B^v + D^v(p_P - p_S). \quad (2.8)$$

$$w_P = A^w_N w_N + A^w_S w_S + A^w_E w_E + A^w_W w_W + B^w + D^w(p_P - p_W). \quad (2.9)$$

$$\phi_P = A^\phi_N \phi_N + A^\phi_S \phi_S + A^\phi_E \phi_E + A^\phi_W \phi_W + B^\phi. \quad (2.10)$$

Here the A coefficients contain mass fluxes, viscosities, diffusion coefficients, etc.; the B coefficients express the effects of convection from the upstream x station, and of source terms (including body forces); the C's are areas across which the fluid flows; and the D's involve areas, mass flow rates, and other quantities. Subscripts D and U in equation (2.6) distinguish downstream (larger-x) values from upstream (smaller-x) ones; but, where neither is subscribed to a variable, the downstream value is meant.

The problem is to solve equations (2.6)–(2.10) simultaneously for all the u's, v's, w's, p's and ϕ's at the downstream x station; the A's, B's, C's and D's can be taken as known, because they can be evaluated with sufficient accuracy from values prevailing at the upstream station. We seek if possible a non-iterative means of solution.

2.3 *An outline of the solution procedure*

The central idea. If the pressures were known,

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there would be little difficulty; for then the momentum equations would be uncoupled, and could be solved individually. They are *not* known in advance, but we can guess the pressures, obtain a first approximation to the velocity field, and then make corrections to the pressure field in such a sense as to bring the velocity field into conformity with the continuity equation.

The confined-flow procedure of the authors' two-dimensional boundary-layer theory has this guess-and-correct feature; but the correction is applied at the next step downstream. This deferred-correction technique could be adopted here; instead however, influenced by the examples of Chorin [14] and Amsden and Harlow [13], we have preferred to make the correction before proceeding to the next step.

The cross-stream pressure and velocities. Let us for the time being assume that we know $(\partial\bar{p}/\partial x)$ and that we have solved equation (2.7) to get the downstream values of u. Now, the next step in our procedure is to obtain a preliminary set of v and w from:

$$v_P^* = A_N^v v_N^* + A_S^v v_S^* + A_E^v v_E^* + A_W^v v_W^* + B^v + D^v(p_P^* - p_S^*). \quad (2.11)$$

$$w_P^* = A_N^w w_N^* + A_S^w w_S^* + A_E^w w_E^* + A_W^w w_W^* + B^w + D^w(p_P^* - p_W^*). \quad (2.12)$$

where the superscript * given to v and w denotes that these are based on an estimated pressure field p^*; usually the upstream values of p are a good estimate.

The starred velocities v^* and w^* will in general not satisfy the continuity equation (2.6), but will produce a net mass source m_P for the point P. This is defined by:

$$m_P \equiv C^v[(\rho v^*)_N - (\rho v^*)_P] + C^w[(\rho w^*)_E - (\rho w^*)_P] + C^u[(\rho u)_{P,U} - (\rho u)_{P,D}]. \quad (2.13)$$

Now our aim is to correct the pressure and velocities so as to annihilate this mass source. For this, we write:

$$p = p^* + p'. \quad (2.14)$$

where p' is the pressure correction. The velocity corrections then follow:

$$v_P = v_P^* + D^v(p'_P - p'_S). \quad (2.15)$$

$$w_P = w_P^* + D^w(p'_P - p'_W). \quad (2.16)$$

It should be noted that the last two are *not* rigorously derived from equations (2.8) and (2.9); we are using approximate forms* of the momentum equations to give us our pressure corrections, just as we did in the two-dimensional confined-flow procedure; and we may expect the practice to suffice here, just as it did before.

The substitution of equations (2.15) and (2.16) into (2.6) gives:

$$p'_P = A_N^p p'_N + A_S^p p'_S + A_E^p p'_E + A_W^p p'_W + B^p, \quad (2.17)$$

where † the A's involve C's, D's and ρ's, and the mass source m_P has been incorporated into B^p. This equation can now be solved to yield the p''s. Thereupon the p's, v's and w's are computed from equations (2.14)–(2.16).

The longitudinal pressure gradient. The foregoing procedure for the calculation of p, v and w was based on the assumption that we knew $(\partial\bar{p}/\partial x)$ and could solve equation (2.7) for u. Here we disclose how $(\partial\bar{p}/\partial x)$ can be obtained. For this purpose, we need to distinguish between external and confined flows. In external flows, $(\partial\bar{p}/\partial x)$ is taken to be the same as the longitudinal pressure gradient prevailing in the irrotational free stream adjacent to the boundary layer. Then the solution of equation (2.7) is straightforward. In confined flow, we regard $(\partial\bar{p}/\partial x)$ as uniform over a cross-section and obtain it from the integral mass-conservation equation in the following manner.

* A correct implication of equation (2.8) would be:

$$v_P = v_P^* + D^v(p'_P - p'_S) + A_N^v(v_N - v_N^*) + A_S^v(v_S - v_S^*) + A_E^v(v_E - v_E^*) + A_W^v(v_W - v_W^*).$$

By dropping the last four terms on the right-hand side of this equation, we get equation (2.15).

† If there are appreciable compressibility effects, care is needed in calculating the densities. This point will not however be elaborated here.

At first, we make an estimate of $(\partial\bar{p}/\partial x)$, which is denoted by $(\partial\bar{p}/\partial x)^*$. This enables us to compute a u^* field from

$$u_P^* = A_N^u u_N^* + A_S^u u_S^* + A_E^u u_E^* + A_W^u u_W^* + B^u + D^u(\partial\bar{p}/\partial x)^*. \quad (2.18)$$

This preliminary velocity field will imply a total mass-flow rate $\Sigma \rho u^* \Delta y \Delta z$ (taken over the duct cross-section) which will in general be different from the true mass-flow rate through the duct, $\dot{m}$, which can be computed directly from the inlet and boundary conditions. The difference can be used to lead us to the correct values of $(\partial\bar{p}/\partial x)$. For this, we write:

$$(\partial\bar{p}/\partial x) = (\partial\bar{p}/\partial x)^* + (\partial\bar{p}/\partial x)', \quad (2.19)$$

$$u_P = u_P^* + D^u(\partial\bar{p}/\partial x)'. \quad (2.20)$$

Since we want

$$\Sigma \rho u \, \Delta y \Delta z = \dot{m}, \quad (2.21)$$

we get, by the substitution of equation (2.20) into (2.21),

$$(\partial\bar{p}/\partial x)' = \frac{\dot{m} - \Sigma \rho u^* \Delta y \, \Delta z}{\Sigma \rho D^u \Delta y \, \Delta z}. \quad (2.22)$$

This gives us the required correction to the longitudinal pressure gradient; so now it is a simple matter to obtain $(\partial\bar{p}/\partial x)$ and u from equations (2.19) and (2.20). The similarity between the equation set (2.11), (2.14), (2.15), and the set (2.18), (2.19), (2.20) should be very obvious. The important difference, however, is that, whereas p' is obtained from the *local* continuity equation, $(\partial\bar{p}/\partial x)'$ is the outcome of the *overall* continuity equation.

Other dependent variables. So far, we have looked at equations (2.6)–(2.9) and obtained the three velocity components and pressure. The equation (2.10) for any other dependent variable ϕ (such as stagnation enthalpy, chemical-species concentration etc.) does not offer any particular difficulty and can be solved straightaway. This completes one forward step.

Solution of the finite-difference equations. In the above description, we referred to "solving" finite-difference equations like equation (2.10). The actual method of solution that we use can be summarized as follows: we employ two sweeps, one in the y and one in the z direction, of the standard tri-diagonal matrix algorithm (TDMA), which is used in the two-dimensional procedure [2] also. Thus, for equation (2.10), ϕ_E and ϕ_W are taken as constants when the sweep is in the y direction, and ϕ_N and ϕ_S are held constant for the sweep in the z direction. More details of this method will be given in Section 3.3.

2.4 *Some general remarks*

The Poisson equation for pressure. At this stage, it will be clear that we obtain the velocity and pressure fields by the solution of the three momentum equations and of the equation (2.17) for the pressure correction p', which is derived from the continuity equation. This equation for p' is just a new form of what is known in the literature as the Poisson equation for pressure. This interpretation may enable the reader to see more clearly why we must treat $(\partial\bar{p}/\partial x)$ differently from $(\partial p/\partial y)$ and $(\partial p/\partial z)$. A general Poisson equation will be elliptic in all the three space co-ordinates and will not allow solution by a marching technique. To be able to march in the x direction, we must treat the term $(\partial^2 p/\partial x^2)$ as known and regard the equation as elliptic in only the y and z co-ordinates. This is precisely why we obtain $(\partial\bar{p}/\partial x)$ *before* the Poisson equation for p' is solved.

The boundary conditions. One of the less obvious but important features of the present method is the ease with which the hydrodynamic boundary conditions can be applied. When we solve for the starred velocity field we can use the actual boundary conditions for velocity, as the starred velocities are expected to be very close to the true velocities. After this is done, the boundary conditions for the pressure correction are also simple: at a wall boundary for example, there will be no velocity correction *at* the boundary, and so the gradient of p' normal to that boundary must be zero; at a boundary adjacent to a free stream on the other hand, the pressure is known, and if p^* is set equal to this pressure, the correction p' at the boundary must be zero. In contrast to the present procedure, the methods that use vorticity as a variable require complicated derivations of the boundary conditions [15].

The non-iterative nature of the procedure. Numerical procedures for solving the partial differential equations in fluid dynamics tend to be iterative for three main reasons: (a) the equations are non-linear; (b) the pressure renders the continuity and momentum equations strongly linked; and (c) a direct solution of the implicit finite-difference equations, even when they are linear, is time-consuming. We have attempted to make the present procedure non-iterative by: (a) the calculation of the A, B, C and D coefficients in the finite-difference equations from values at the upstream station; (Thus, we "force" the equations to be linear); (b) the use of approximate forms of momentum equations (equations (2.15), (2.16) and (2.20)); and (c) the solution of the finite-difference equations by the two sweeps of the TDMA. It is true that these three "tricks" introduce some errors in our solution compared to a solution produced by a fully iterative procedure. But, firstly, these errors are of the same kind as the "truncation" errors in any finite-difference procedure and hence can be reduced to an acceptable level by the use of small forward steps; and secondly, it is possible for us, at the end of each forward step, to calculate the error in satisfying each conservation equation (these can be considered as mass or momentum sources which our numerical approximations have introduced) and then to make a corresponding correction at the next step downstream. Thus, by leaving errors which can be detected and, if necessary, corrected for, we enjoy the benefits of a non-iterative procedure without serious penalty.

We hope by now to have conveyed to the reader the essential features of our calculation procedure. The actual algebraic details remain to be given. It is to this matter that we now turn.

3. SOME DETAILS OF THE CALCULATION PROCEDURE

3.1 *Restrictions*

The general calculation procedure described so far is restricted only by those conditions which define parabolic flows, and which are described in Section 1.2. However, the algebraic details of the general procedure with various types of boundary conditions, grid systems, auxilliary information, etc. will be quite lengthy and tedious to report here. For this reason, we shall present the equations for a uniform-property laminar flow and give only the important details. The remaining details are either so straightforward that the reader could work them out himself, or are similar to the corresponding features of our two-dimensional procedure [2]. We shall use a Cartesian coordinate system xyz.

3.2 *The finite-difference equations*

The differential equation considered. For a laminar uniform-property flow, equation (2.5) takes the form:

$$\underset{(1)}{\frac{\partial}{\partial x}(\rho u\phi)} + \underset{(2)}{\frac{\partial}{\partial y}(\rho v\phi)} + \underset{(3)}{\frac{\partial}{\partial z}(\rho w\phi)} = \underset{(4)}{\Gamma\frac{\partial^2\phi}{\partial y^2}} + \underset{(5)}{\Gamma\frac{\partial^2\phi}{\partial z^2}} + \underset{(6)}{S_\phi}. \quad (3.1)$$

where Γ is the transport property such as viscosity. When ϕ stands for a velocity component, the differential equation has the same form except that a pressure-gradient term appears on the right-hand side. (This term should be written separately, and not included in S_ϕ, as we treat the pressure as an unknown.) Therefore, it will be sufficient to describe here how equation (3.1) is transformed into a finite-difference equation.

Some basic decisions. We transform equation (3.1) into a finite-difference equation by integrating it over the control volume shown in Fig. 4 by dotted lines. Figure 5 gives more details of the yz face of the control volume. The points n, s, e, w are the midpoints of the lines PN, PS, PE and PW respectively. (The "boomerangs" in Fig. 3 have disappeared in Figs. 4 and 5; there the points n, s, e, w have been introduced,

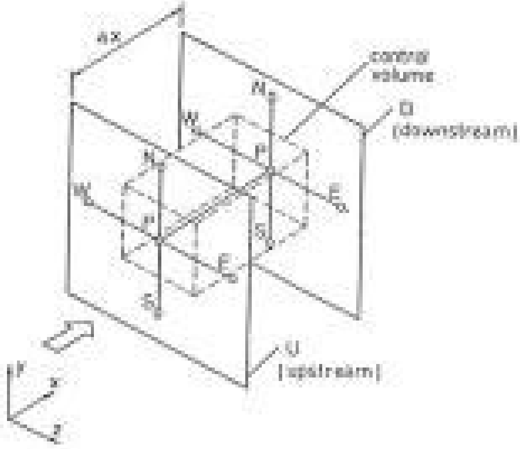

FIG. 4. The control volume used to obtain the finite-difference equation.

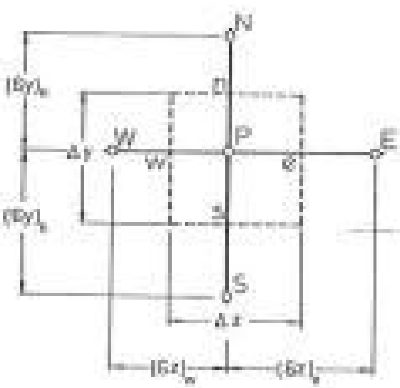

FIG. 5. The yz face of the control volume.

which enable us to present the algebraic details more precisely.) We make the following assumptions about the variation of ϕ between the grid points:

(a) In the x direction ϕ varies in a stepwise manner; i.e. the *downstream* ($x = x_D$) values of ϕ are supposed to prevail over the interval from x_U to x_D *except at* x_U. This makes our finite-difference scheme a fully-implicit one.

(b) For the calculation of the x-direction convection and of source terms that may depend on ϕ, the variation of ϕ in the yz plane is also taken to be stepwise. Thus, in the yz plane the value of ϕ is assumed to remain uniform and equal to ϕ_P over the dotted rectangle (Fig. 5) surrounding the point P and to change sharply to ϕ_N, ϕ_S, ϕ_E or ϕ_W outside the rectangle.

(c) For the cross-stream convection from the xy and xz faces of the control volume, the value of ϕ convected is taken to be the arithmetic mean of the ϕ values on either side of that face, except when this practice is altered by the "high-lateral-flux modification" mentioned below. Thus we use a convenient combination of the central-difference and upwind-difference formulae for the first-order derivatives.

(d) For diffusion across the xy and xz faces of the control volume, we assume that ϕ varies linearly between grid points, except when the high-lateral-flux modification dictates otherwise.

Details of the main finite-difference equation. When the above-mentioned decisions are taken, it is a simple matter to obtain the finite-difference equation by integration of equation (3.1) over the control volume. We get:

$$\underbrace{F_D\phi_P - F_U\phi_{P,U}}_{(1)} + \underbrace{L^y_n(\phi_N + \phi_P) - L^y_s(\phi_S + \phi_P)}_{(2)}$$
$$\underbrace{+ L^z_e(\phi_E + \phi_P) - L^z_w(\phi_W + \phi_P)}_{(3)}$$
$$\underbrace{= T^y_n(\phi_N - \phi_P) - T^y_s(\phi_P - \phi_S)}_{(4)}$$
$$\underbrace{+ T^z_e(\phi_E - \phi_P) - T^z_w(\phi_P - \phi_W)}_{(5)}$$
$$\underbrace{+ S_U + S_P\phi_P,}_{(6)} \quad (3.2)$$

where the numbers in the parentheses indicate the corresponding terms in equation (3.1), and the new symbols are defined as follows:

$$\left.\begin{aligned}
F_U &\equiv \frac{(\Delta y)(\Delta z)}{\Delta x}(\rho u)_{P,U}\\
L^y &\equiv \frac{(\Delta z)}{2}(\rho v)_U\\
L^z &\equiv \frac{(\Delta y)}{2}(\rho w)_U\\
F_D &\equiv F_U - 2L^y_n + 2L^y_s - 2L^z_e + 2L^z_w\\
T^y &\equiv \frac{\Gamma(\Delta z)}{\delta y}\\
T^z &\equiv \frac{\Gamma(\Delta y)}{\delta z}\\
(S_U + S_P\phi_P) &\equiv S_{\phi,P}(\Delta y)(\Delta z)
\end{aligned}\right\} \quad (3.3)$$

Rearranging the terms, we get:

$$\phi_P = A_N\phi_N + A_S\phi_S + A_E\phi_E + A_W\phi_W + B, \quad (3.4)$$

where

$$\left.\begin{aligned}
A_N &\equiv A'_N/A'_P,\\
A_S &\equiv A'_S/A'_P,\\
A_E &\equiv A'_E/A'_P,\\
A_W &\equiv A'_W/A'_P,\\
B &\equiv B'/A'_P;\\
A'_N &\equiv T^y_n - L^y_n,\\
A'_S &\equiv T^y_s + L^y_s,\\
A'_E &\equiv T^z_e - L^z_e,\\
A'_W &\equiv T^z_w + L^z_w,\\
B' &\equiv F_U\phi_{P,U} + S_U,\\
A'_P &\equiv A'_N + A'_S + A'_E + A'_W + F_U - S_P.
\end{aligned}\right\} \quad (3.5)$$

The high-lateral-flux modification. When the lateral flow (denoted by the symbol L) is large, some of the coefficients A_N, A_S, A_E, A_W can become negative; this event leads to physically unrealistic results. The cure is a simple one and is discussed at length in [2], where it is given the name "the high-lateral-flux modification". Here we merely state that the modification consists of replacing all the T's by $\tilde{T}$'s defined by:

$$\tilde{T} \equiv (\tfrac{1}{2})\{T + |L| + |T - |L||\}. \quad (3.6)$$

where the T and L should be the corresponding ones (e.g. T^y_n with L^y_n). It should be noted that this modification becomes "active" only when $|L| > T$; T itself is always positive.

Finite-difference equations for velocity components. As mentioned earlier, the difference equations for u, v and w will be similar to equation (3.4) except for an additional pressure term. In deriving the equations for the cross-stream velocities v and w, we must note that, since v and w have "staggered" storage locations, they require different control volumes. The actual details, however, will not be given here.

Finite-difference equation for pressure correction. If we write equations (2.15) and (2.16) as:

$$\left.\begin{aligned}
v_n &= v^*_n + D^v_n(p'_N - p'_P),\\
v_s &= v^*_s + D^v_s(p'_P - p'_S),\\
w_e &= w^*_e + D^w_e(p'_E - p'_P),\\
w_w &= w^*_w + D^w_w(p'_P - p'_W),
\end{aligned}\right\} \quad (3.7)$$

the continuity equation written for the control volume shown in Fig. 4 becomes:

$$\begin{aligned}
F_U\left\{\frac{u_{P,D}}{u_{P,U}} - 1\right\} &+ 2L^{y*}_n - 2L^{y*}_s + 2L^{z*}_e\\
&- 2L^{z*}_w + \rho\Delta z\, D^v_n(p'_N - p'_P) + \rho\Delta z\, D^v_s(p'_S\\
&- p'_P) + \rho\Delta y\, D^w_e(p'_E - p'_P)\\
&+ \rho\Delta y\, D^w_w(p'_W - p'_P) = 0. \quad (3.8)
\end{aligned}$$

The superscript * on the L's denotes that these are calculated from the starred velocity components. Now it is a mere matter of rearrangement to get equation (2.17).

3.3 *Solution of the finite-difference equations*

The double sweep. The finite-difference equations like (3.4) can be solved by the successive use of the TDMA in the y and z directions. For the y-direction sweep, we write:

$$\phi_P^{\rm I} = A_N \phi_N^{\rm I} + A_S \phi_S^{\rm I} + (A_E \phi_{E,U} + A_W \phi_{W,U} + B), \quad (3.9)$$

where the expression in the parentheses is known and the TDMA can be applied. The superscript I denotes the values obtained from this first phase of solution. The second phase, namely the z-direction sweep, is the solution of:

$$\phi_P^{\rm II} = A_E \phi_E^{\rm II} + A_W \phi_W^{\rm II} + (A_N \phi_N^{\rm I} + A_S \phi_S^{\rm I} + B) \quad (3.10)$$

in a similar manner.

Remarks. It is true that the above procedure does not give us an exact solution of the finite-difference equations; but its use is advocated on the following considerations:

(1) It can be easily seen that, when the y-direction coefficients A_N, A_S are much smaller or much larger in magnitude than A_E, A_W, the above procedure does give a nearly correct solution.

(2) When the forward step Δx is small, the equation is dominated by B which contains the upstream value $\phi_{P,U}$; then the use of slightly approximate values of ϕ_N, ϕ_S, ϕ_E, ϕ_W introduce a very small error in ϕ_P.

(3) The last remark applies to all the finite-difference equations except the one for the pressure correction, which does not have an "upstream convection" term. For the pressure-correction equation, therefore, it may be worth-while obtaining greater accuracy by *repeating* the double sweep a few times. Usually about three executions of the double sweep are sufficient.

(4) Thus, to reduce the error resulting from the TDMA-double-sweep method of solution, we can use one or more of the following devices:

(i) use smaller forward step;
(ii) repeat the double sweep a small number of times;
(iii) calculate the error at the end of the forward step and correct for it during the next step.

3.4 *Some miscellaneous matters*

Many details of the calculation procedure still remain to be reported. Here we merely draw attention to a few points and state that these can be handled by means similar to those in the authors' two-dimensional procedure [2].

The turbulent boundary layer. When the flow is turbulent rather than laminar, the same calculation procedure is to be used except that the laminar viscosity and other transport properties are to be replaced by "effective" transport coefficients given by a "turbulence model".

The specified-flux boundary. When at a wall boundary the heat flux (rather than the temperature) is specified, the finite-difference equation for a control volume adjacent to that boundary must be rewritten in such a way that the coefficient of the boundary temperature is zero.

The wall functions. Often the variations of the dependent variables are quite steep near a wall boundary and therefore the diffusion flux at the wall cannot be accurately obtained from a linear-profile assumption for ϕ. In such cases, one can employ a function (called the wall function in [2]) for the flux at the wall; this takes into account the non-linearity of the ϕ profile resulting from pressure gradient, mass transfer, transport-property variation etc.

Adjusting grids. In this paper, we have used a Cartesian coordinate system throughout; but it is possible to employ other coordinate systems which may be convenient for particular problems. For example, the flow in a duct of elliptic cross-section can be conveniently calculated on a curvilinear orthogonal coordinate system in the cross-stream plane. For external boundary layers, it is profitable to use a grid (as in [2]), which expands or contracts as the boundary-layer thickness increases or decreases.

4. AN APPLICATION OF THE CALCULATION PROCEDURE

4.1 *Statement of the problem*

Here we illustrate the use of the present method by applying it to the developing flow and heat transfer in a duct of square cross-section with a laterally-moving wall, as shown in Fig. 6. This flow situation is found in screw extruders, bearing lubricators, membrane oxygenators etc. Further, in regions of fully-developed flow, the cross-stream velocity and pressure fields are identical to those in a steady

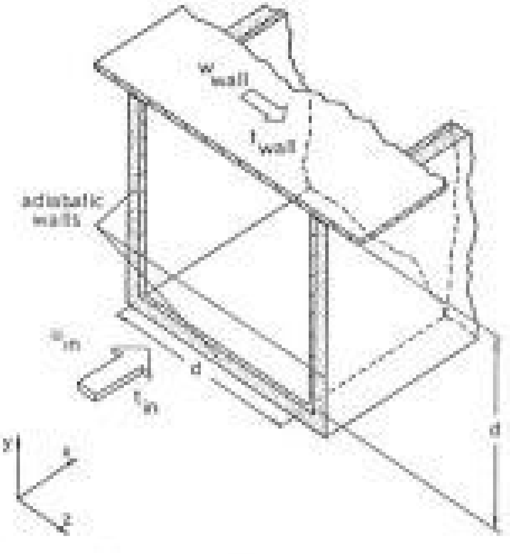

FIG. 6. The flow situation considered.

two-dimensional flow in a square cavity with a moving wall.* The latter problem has been analysed by many authors (for example, [16–18]), and we have their solutions for comparison.

The flow is regarded as laminar, and the fluid properties as uniform. At the inlet, the velocity, pressure and temperature are taken to be uniform over the cross-section. The temperature of the moving wall is held at a fixed value, whereas the other three walls are considered aidabatic. Buoyancy effects are neglected.

* This can be easily understood if we note that, in the fully-developed region, all the velocity components cease to vary with x. Then the cross-stream velocities v and w are governed by precisely the same equations as those for a two-dimensional flow.

4.2 Details of the computation

The computations were performed on an IBM 7094 computer. A uniform rectangular grid of 16 × 16 nodes was used for all runs except those which were made to examine the effect of the grid size. Each forward step took 2 s of computer time. About 100 forward steps were necessary to attain the fully-developed situation.

4.3 Results

The mean pressure. Figure 7 shows the variation of the mean pressure with the distance along the duct for various velocities of the moving wall. For the case of the stationary wall, our predictions are compared with the experimental data of [19]; the agreement can be seen to be very good.

The effect of the grid size. In Fig. 8 are plotted the predictions of mean-pressure variation for various grid sizes. As can be expected, the successive refinement of the grid takes us asymptotically towards the correct solution. We can conclude that a 16 × 16 grid gives us a sufficiently accurate solution for this problem.

The velocity field. Figure 9 shows the variation of the maximum longitudinal velocity with the distance along the duct. Once again, the predictions for zero wall velocity are compared with the experimental data of [20] and the agreement is good. Figures 10 and 11 refer to the fully-developed (large-x) region of the flow. In Fig. 10 are presented the contours of the longitudinal velocity for various velocities of the moving wall. Figure 11 compares the variation of a cross-stream velocity (along a centre-line of the cross-section) with the numerical results of Burggraf [16], who solved the steady two-dimensional square-cavity problem. Once again the agreement is very satisfactory.

The temperature field. Figure 12 shows how the bulk temperature of the fluid rises with the longitudinal distance. As can be expected, the higher the wall velocity, the faster is the rise of the bulk temperature. Figure 13 shows the effect of the Prandtl number on the bulk-

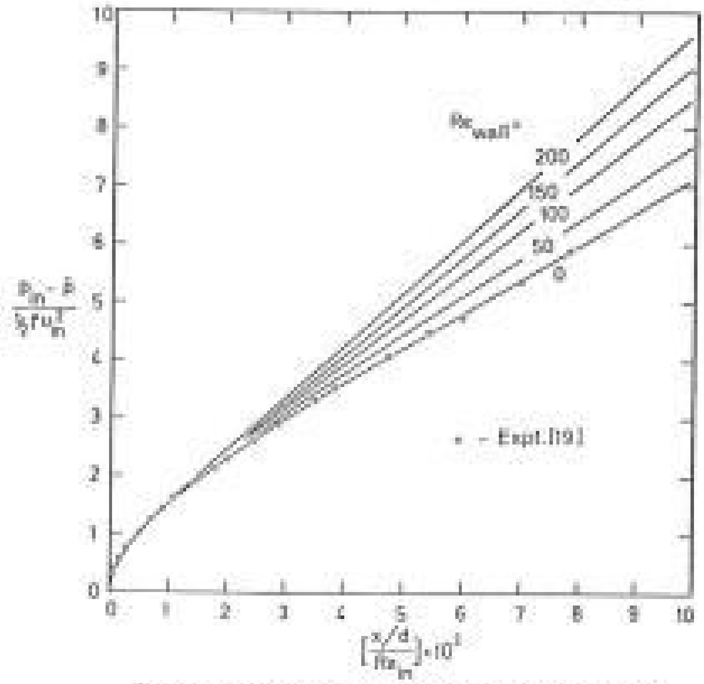

FIG. 7. Variation of mean pressure with longitudinal distance.

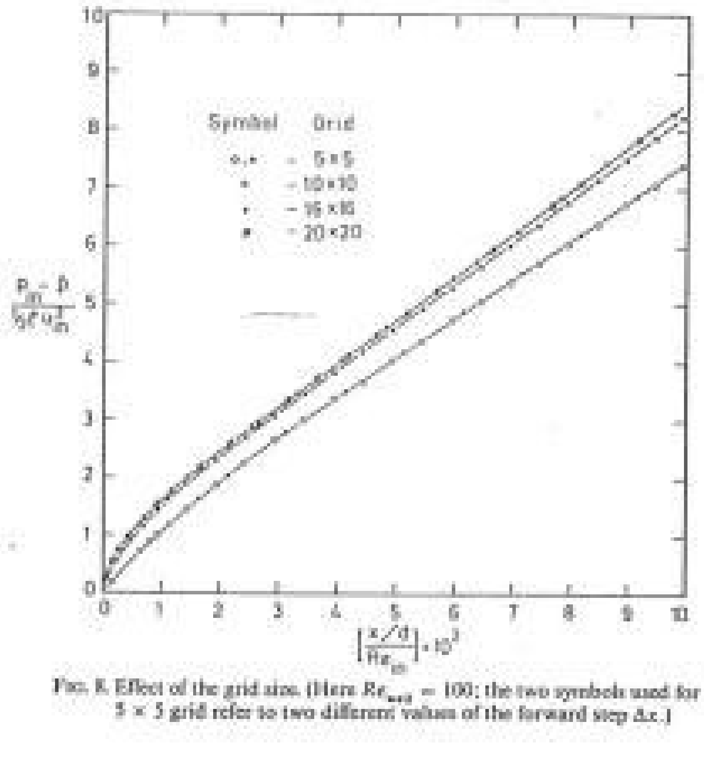

FIG. 8. Effect of the grid size. (Here Re_{wall} = 100; the two symbols used for the 5 × 5 grid refer to two different values of the forward step Δx.)

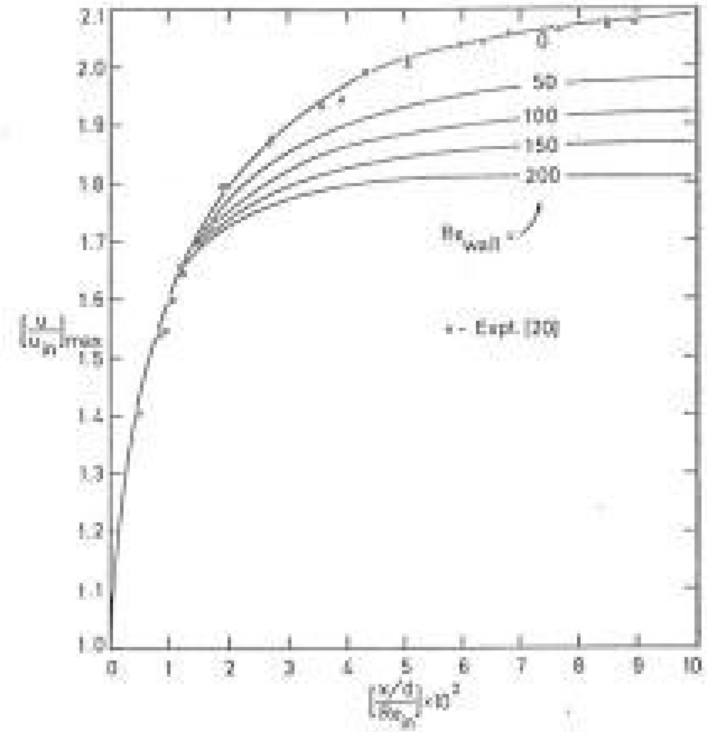

FIG. 9. Variation of maximum longitudinal velocity.

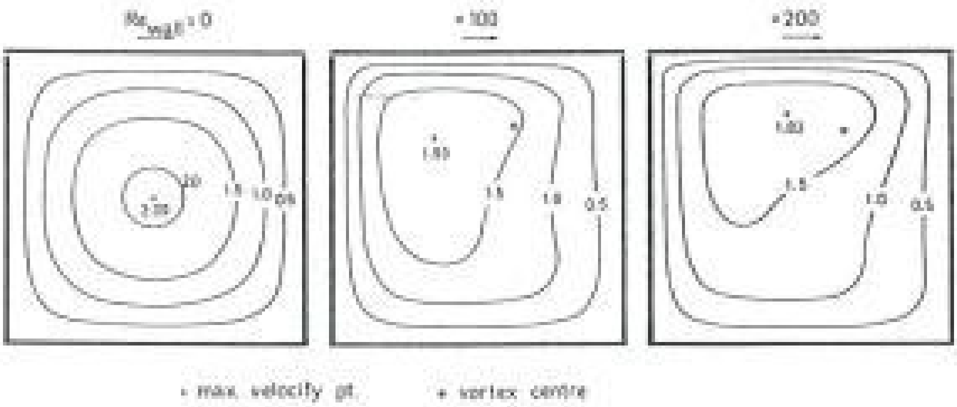

FIG. 10. Longitudinal-velocity contours in the fully-developed region.

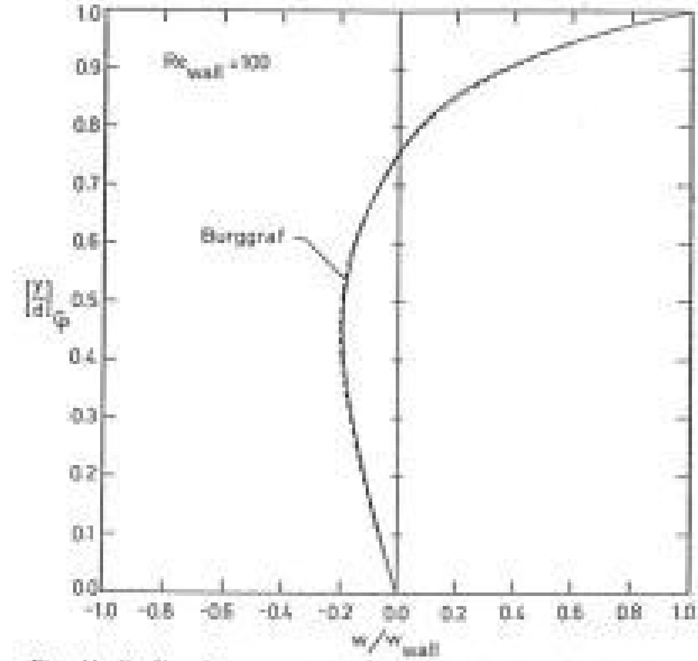

FIG. 11. Profile of a cross-stream velocity in the fully-developed region.

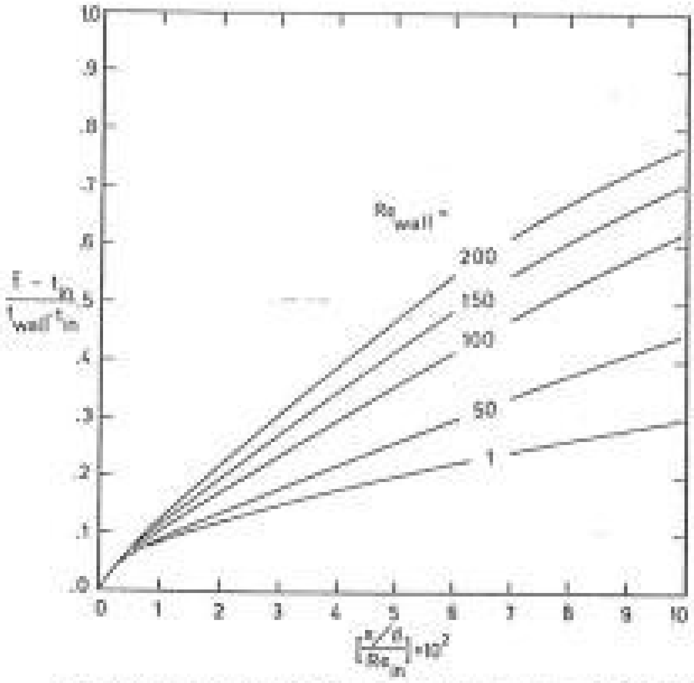

FIG. 12. Variation of bulk temperature for various wall velocities ($Pr = 1$).

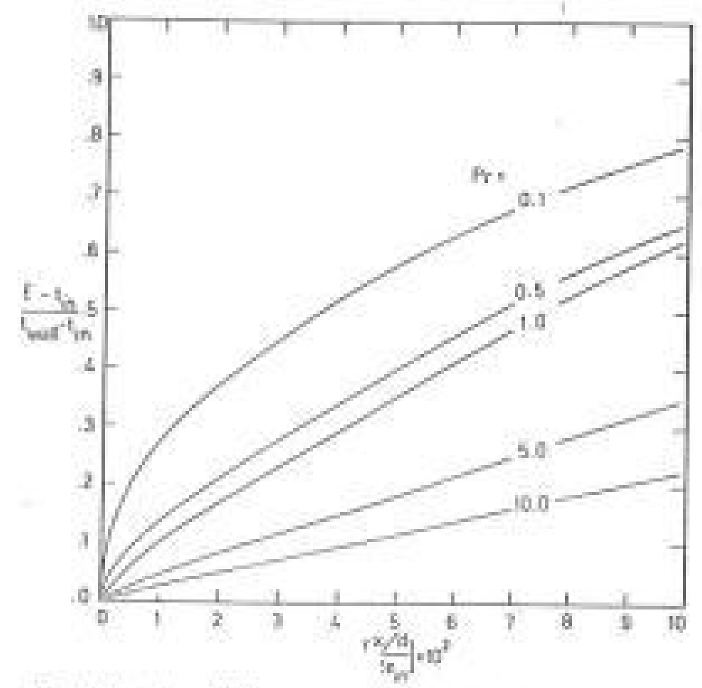

FIG. 13. Variation of bulk temperature for various Prandtl numbers ($Re_{wall} = 100$).

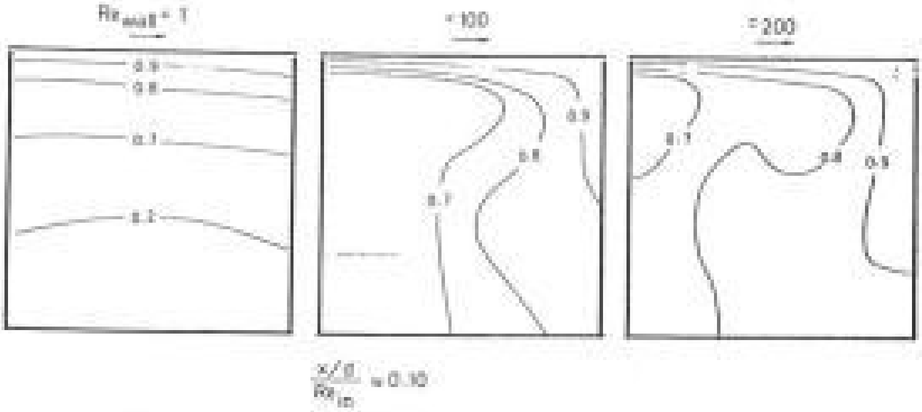

FIG. 14. Contours of $(t - t_{in})/(t_{wall} - t_{in})$ for various wall velocities ($Pr = 1$).

temperature development. The temperature distribution in the cross-stream plane is presented in Fig. 14 in the form of contours; it can be seen how the swirl induced by the moving wall distorts the temperature field.

5. CONCLUDING REMARKS

(1) The present paper has described a generally applicable, accurate and economical method for calculating heat, mass and momentum transfer in three-dimensional parabolic flows.

(2) The uncoupling of the longitudinal and cross-stream pressure gradients is an important feature of the method; it is essential for making the equations parabolic.

(3) The non-iterative nature of the method derives from the use of upstream convection fluxes, from the explicit corrections of pressure and velocity, and from the double-sweep-IDMA solution of the finite-difference equations.

(4) The procedure described here shares many useful features with the present authors' two-dimensional procedure [2].

(5) Various applications of the present procedure are in progress, and will be reported elsewhere. Further advances in the prediction of three-dimensional parabolic flows would come from the development of the models for turbulence, radiation and chemical reaction.

ACKNOWLEDGEMENTS

The assistance of Mr. D. Sharma in some of the computations is gratefully acknowledged. The authors have benefited also from the prior and simultaneous work of colleagues and students at Imperial College, especially Drs. L. S. Caretto, R. M. Curr and A. D. Gosman, and Messrs. D. Sharma and D. G. Tatchell.

REFERENCES

1. Proceedings of the AFOSR-IFP-Stanford Conference, Thermoscience Division, Stanford University (1968).
2. S. V. Patankar and D. B. Spalding, *Heat and Mass Transfer in Boundary Layers*, Second Edition. Intertext Books, London (1970).
3. G. S. Raetz, A method of calculating three-dimensional laminar boundary layers of steady compressible flows, Northrop Aircraft Inc., Rep. No. NAI-58-73 (BLC-144) (1957).
4. M. G. Hall, A numerical method for calculating steady three-dimensional laminar boundary layers, RAE Tech. Rep. 67145 (June 1967).
5. H. A. Dwyer, Solution of a three-dimensional boundary layer flow with separation, *AIAA Jl* 6, 1336 (1968).
6. T. K. Fannelop, A method of solving the three-dimensional laminar boundary-layer equations with application to a lifting re-entry body, *AIAA Jl* 6, 1075 (1968).
7. J. F. Nash, The calculation of three-dimensional turbulent boundary layers in compressible flow, *J. Fluid Mech.* 37, 625 (1969).
8. E. Krause and E. H. Hirschel, Exact numerical solutions for three-dimensional boundary layers, Second International Conference on Numerical Methods in Fluid Dynamics, University of California, Berkeley (Sept. 1970).
Also see: E. Krause, E. H. Hirschel and Th. Bothmann, Die numerische Integration der Bewegungsgleichungen drei-dimensionaler laminarer kompressibler Grenzschichten, Fachtagung Aerodynamik, Berlin, DGLR Fachbuchreihe Band 3, Braunschweig (1968).
9. K. C. Wang, Three-dimensional boundary layer near the plane of symmetry of a spheroid at incidence, *J. Fluid Mech.* 43, 187 (1970).
10. J. A. Miller, Laminar incompressible flow in the entrance region of ducts of arbitrary cross section, *J. Engng Power* 113 (Jan. 1971).
11. F. H. Harlow, Numerical methods for fluid dynamics, an annotated bibliography, Los Alamos Scientific Laboratory, LA-4281 (1969).
12. F. H. Harlow and J. E. Welch, Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface, *Physics Fluids* 8, 2182 (1965).
13. A. A. Amsden and F. H. Harlow, The SMAC method: a numerical technique for calculating incompressible fluid flows, Los Alamos Scientific Laboratory, LA-4370 (Feb. 1970).
14. A. J. Chorin, Numerical solution of the Navier-Stokes equations, *Maths Comput.* 22, 745 (1978).
15. L. S. Caretto, R. M. Curr and D. B. Spalding, Two numerical methods for three-dimensional boundary layers, Imperial College, Mech. Eng. Dept. Rep. EF/TN/A/40 (July 1971).
16. O. R. Burggraf, Analytical and numerical studies of structure of steady separated flows, *J. Fluid Mech.* 24, Pt. 2, 113–51 (1966).
17. A. K. Runchal, D. B. Spalding and M. Wolfshtein, Numerical solution of the elliptic equation for transport of vorticity, heat, and matter in two-dimensional flow, *Physics Fluids, Suppl.* 11, pp. 11–21 (1969).
18. L. F. Donovan, A numerical solution of unsteady flow in a two-dimensional square cavity, *AIAA Jl* 8, 534 (1970).
19. G. S. Beavers, E. M. Sparrow and R. A. Magnuson, Experiments on hydrodynamically developing flow in rectangular ducts of arbitrary aspect ratio, *Int. J. Heat Mass Transfer* 13, 689 (1970).
20. R. J. Goldstein and D. K. Kreid, Measurement of laminar flow development in a square duct using a Laser-Doppler flowmeter, *J. Appl. Mech.* 813 (1967).

UNE METHODE DE CALCUL DU TRANSFERT DE CHALEUR, DE MASSE ET DE QUANTITE DE MOUVEMENT DANS LES ECOULEMENTS PARABOLIQUES TRIDIMENSIONNELS

Résumé—Une méthode numérique générale est présentée pour le calcul des processus de transport dans des écoulements tridimensionnels caractérisés par la présence d'une coordonnée pour laquelle les influences physiques sont sensibles dans une seule direction. De tels écoulements donnent lieu à des équations aux dérivées partielles paraboliques et ainsi peuvent être appelés écoulements paraboliques tridimensionnels. La procédure peut être considérée comme une méthode de couche limite mais on remarque que, contrairement à des méthodes antérieurement publiées sous ce nom, elle tient entièrement compte de la diffusion transversale de quantité de mouvement etc. et de la variation de pression dans le plan perpendiculaire à l'écoulement. Le champ de pression est déterminé par: premièrement le calcul d'un champ de vitesse intermédiaire basé sur un champ de pression estimé; ensuite par l'obtention de corrections appropriées de façon à satisfaire l'équation de continuité. Pour illustrer la méthode, des calculs sont présentés pour le développement d'un écoulement laminaire et du transfert thermique dans un conduit carré avec une paroi mobile latéralement.

EIN RECHENVERFAHREN FÜR WÄRME-, STOFF- UND IMPULSÜBERTRAGUNG IN DREIDIMENSIONALEN PARABOLISCHEN STRÖMUNGEN

Zusammenfassung—Es wird ein allgemeines numerisches Verfahren gezeigt für die Berechnung der Transportvorgänge in dreidimensionalen Strömungen, die dadurch gekennzeichnet sind, dass in einer Koordinate physikalische Einflüsse nur in einer Richtung wirken. Solche Strömungen führen zu parabolischen Differentialgleichungen, daher kann man von dreidimensionalen parabolischen Strömungen sprechen. Das Rechenverfahren kann als Grenzschichtmethode betrachtet werden, wobei allerdings im Unterschied zu früher veröffentlichten Methoden mit diesem Namen der Impulstransport u.ä. senkrecht zur Strömung und auch die Druckänderung in der Ebene senkrecht zur Strömung voll berücksichtigt sind. Das Druckfeld wird wie folgt bestimmt: Die erste Berechnung eines mittleren Geschwindigkeitsfeldes beruht auf einem angenommenen Druckfeld; dann werden passende Korrekturen ermittelt, so dass die Kontinuitätsgleichung erfüllt wird. Um das Rechenverfahren zu demonstrieren, werden Berechnungen für den Einlaufvorgang einer laminaren Strömung mit Wärmeübergang in einem quadratischen Kanal mit einer querbewegten Wand angeführt.

РАСЧЕТ ПЕРЕНОСА ТЕПЛА, МАССЫ И ИМПУЛЬСА В ТРЕХМЕРНЫХ ПАРАБОЛИЧЕСКИХ ПОТОКАХ

Аннотация—Представлен общий конечно-разностный метод расчета процессов переноса в трехмерных течениях, характеризующихся наличием одного преимущественного направления изменения, на котором физические эффекты сказываются только в характеристике переноса. Эти течения описываются параболическими дифференциальными уравнениями и могут быть названы трехмерными параболическими течениями. Представленный метод можно рассматривать как метод расчета пограничного слоя, который в отличие от известных ранее методов полностью учитывает поперечный перенос количества движения и т.д., а также изменение давления в плоскости, нормальной к потоку. Предварительно поле давления определяется путем расчета поля скорости по наперед заданному полю давления с последующим введением соответствующих поправок с тем, чтобы удовлетворить уравнению неразрывности. Для иллюстрации приводятся расчеты развивающегося ламинарного течения и переноса тепла в трубе квадратного сечения с движущейся горизонтальной стенкой.

APPENDIX 'D'

Two Calculation Procedures for Steady, and Three-Dimensional Flows with Recirculation

by

L S Caretto, A D Gosman, S V Patankar and D B Spalding

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TWO CALCULATION PROCEDURES FOR STEADY, THREE-DIMENSIONAL FLOWS WITH RECIRCULATION

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ABSTRACT

Two procedures are described for solving the Navier-Stokes equations for steady, fully three-dimensional flows: both are extensions of earlier methods devised for three-dimensional boundary layers, and have the following common features: (i) the main dependent variables are the velocities and pressure; (ii) the latter are computed on a number of staggered, interlacing grids, each of which is associated with a particular variable; (iii) a hybrid central-upwind difference scheme is employed; and (iv) the solution algorithms are sufficiently implicit to obviate the need to approach the steady state via the time evolution of the flow, as is required by wholly explicit methods.

The procedures differ in their manner of solving the difference equations. The SIVA (for SImultaneous Variable Adjustment) procedure, which is fully-implicit, uses a combination of algebraic elimination and point-successive substitution, wherein *simultaneous* adjustments are made to a point pressure, and the six surrounding velocities, such that the equations for mass and (linearised) momentum are locally satisfied.

The SIMPLE (for Semi-Implicit Method for Pressure-Linked Equations) method proceeds in a successive guess-and-correct fashion. Each cycle of iteration entails firstly the calculation of an intermediate velocity field which satisfies the linearised momentum equations for a guessed pressure distribution: then the mass conservation principle is invoked to adjust the velocities and pressures, such that all of the equations are in balance.

By way of an illustration of the capabilities of the methods, results are given of the calculation of the flow of wind around a building, and the simultaneous dispersal of the effluent from a chimney located upstream.

1. INTRODUCTION

1.1 Objectives of the present research. We are here concerned with prediction methods for that class of convective-flow phenomena which are steady, recirculating, low-speed and three-dimensional: the majority of the practically-important flow situations encountered in industrial, environmental, physiological and other fields are of this kind. Two calculation procedures for such flows will be described: both proceed by way of finite-difference solution of the Eulerian partial-differential equations for the conservation of mass, momentum, energy and other properties; and both employ the velocities and pressure as the main hydrodynamic variables.

1.2 Relation to previous work. Although there exist a number of finite-difference procedures which could, in principle, be used for the present class of problems, none appear to be well-suited for this purpose. Thus, for example, nearly all of the available methods attempt to follow the time evolution of the flow in arriving at the steady-state solution. When however the latter is the *only* feature of interest, this is usually needlessly expensive, especially when an *explicit* formulation is employed.

The procedures to be described here contain a number of innovations, allowing particularly economical routes to the steady state: they also however incorporate many known features including:- the displaced grids for velocity and pressure employed by Harlow and Welch (1965); the concept of a guess-and-correct procedure for

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the velocity field, used by Amsden and Harlow (1970) and Chorin (1968); and the implicit calculation of velocities, along the lines of the Fracht (1970) version of the Harlow-Welch (1965) procedure. Additional guidance in the formulation of the new procedures has been derived from earlier work by the authors and their colleagues on methods for two-dimensional flows (Patankar and Spalding, 1970; Gosman et al., 1969), and three-dimensional boundary layers (Patankar and Spalding, 1972a; Caretto et al., 1972).

1.3 Contents of the paper. Section 2 of the paper is devoted to the description of the two procedures, code-named SIMPLE and SIVA. Because the point of departure between the two methods is in the manner of solving the finite-difference equations, the latter are described first; then details are given of the individual solution paths.

In Section 3, we provide a summary of the experience gained from application of the procedures to a variety of test cases. Then, by way of a demonstration, we present the results of a computer simulation of the flow of wind past a building, and the simultaneous dispersal of the effluent from a chimney located upwind of the building. Finally, in Section 4 we present our conclusions about the relative merits of the two procedures, and the prospects for further development.

2. ANALYSIS

2.1 The equations to be solved. The mathematical problem may be compactly expressed, with the aid of Cartesian tensor notation, in terms of the following set of differential equations:

$$\partial(\rho u_i)/\partial x_i = 0 \quad ; \qquad (1)$$

$$\partial(\rho u_i u_j)/\partial x_i - \partial(\mu_{eff}\,\partial u_j/\partial x_i)/\partial x_i + \partial P/\partial x_j - s_j = 0 \quad ; \qquad (2)$$

$$\partial(\rho u_i \phi)/\partial x_i - \partial(\Gamma_{\phi,eff}\,\partial\phi/\partial x_i)/\partial x_i - s_\phi = 0 \quad ; \qquad (3)$$

which express the laws of conservation of mass, momentum and a scalar property ϕ respectively. Here the dependent variables are the (time-average) values of: the velocities u_j; the pressure P; and ϕ, which stands for such scalar quantities as enthalpy, concentration, kinetic energy or dissipation rate of turbulence (Launder and Spalding, 1971) and radiation flux (Spalding, 1972a) etc. The symbols s_j and s_ϕ stand for additional sources (or sinks) associated with such phenomena as natural convection, chemical reaction and non-uniformity of transport coefficients, while ρ, μ_{eff} and $\Gamma_{\phi,eff}$ are respectively the density, viscosity and exchange coefficient for ϕ. The subscript 'eff' appended to the latter two indicates that, for turbulent flows, they are sometimes ascribed 'effective' values, deduced from turbulence quantities.

2.2 Finite-difference equations

(a) Grid and notation. The staggered-grid system employed for both methods is depicted in Fig. 1: this shows only the xy plane, but the treatment in the other planes follows identical lines.

The intersections of the solid lines mark the grid nodes, where all variables except the velocity components are stored. The latter are stored at points which are denoted by the arrows and located mid-way between the grid intersections. A considered node and its immediate neighbours are denoted by the subscripts P, x+, x-, y+, y-, z+ and z-: the significance of these can be perceived from Fig. 1. The velocities are similarly referenced, with the convention that P (and each of the other subscripts) now refers to a cluster of variables, as indicated in the diagram.

(b) Differencing practices. Attention will first be focussed on the differential conservation equation (3) for a scalar property ϕ. A difference equation relating ϕ_P to the surrounding ϕ's is obtained by integration of (3) over the control volume enclosing P, with the aid of flux expressions derived from one-dimensional flow theory. Some details will now be given.

We represent the net x-direction convection and diffusion of ϕ through the control volume (Fig. 2) by:

$$C^{\phi}_{x+}(\phi_{x+} - \phi_P) + C^{\phi}_{x-}(\phi_{x-} - \phi_P) \tag{4}$$

where, e.g.:

$$C^{\phi}_{x+} = \begin{cases} 0, \text{ when } F_{x+} > D_{x+} & ; \\ -2F_{x+}, \text{ when } F_{x+} < -D_{x+} & ; \\ D_{x+} - F_{x+}, \text{ in all other circumstances.} \end{cases}$$

$$F_{x+} \equiv \dot{m}''_{+} A_x/2 \quad ; \qquad D_{x+} \equiv \Gamma_{\phi+} A_x/\delta x_{+} \quad ;$$

and $\dot{m}''_{+}$, A_x and $\Gamma_{\phi+}$ respectively stand for the mass flux, cross-sectional area and average exchange coefficient at the boundary in question. The other quantities in (4) are similarly defined.

The above expression may be regarded as a hybrid of central- and upwind-difference schemes, in that it reduces to the former when the ratio $|F/D|$ (a local Peclet number) is less than unity; and it yields the large- (F/D) asymptote of the latter for $|F/D|$ greater than unity. The hybrid scheme has the advantages of being more accurate over a wide range of F/D (Spalding, 1972b; Runchal, 1970), than either of its components, and of yielding a diagonally-dominant matrix of coefficients for all F/D.

(c) The difference equations. When the fluxes in the y and z directions are expressed in a similar manner, the resultant finite-difference equation is:

$$C^{\phi}_P \phi_P = C^{\phi}_{x+}\phi_{x+} + C^{\phi}_{x-}\phi_{x-} + C^{\phi}_{y+}\phi_{y+} + C^{\phi}_{y-}\phi_{y-} + C^{\phi}_{z+}\phi_{z+} + C^{\phi}_{z-}\phi_{z-} + S^{\phi} \quad ; \tag{5}$$

where S^{ϕ} represents the integral of the source s_{ϕ} over the control volume and:

$$C^{\phi}_P \equiv C^{\phi}_{x+} + C^{\phi}_{x-} + C^{\phi}_{y+} + C^{\phi}_{y-} + C^{\phi}_{z+} + C^{\phi}_{z-} \quad .$$

The treatment of the momentum equations is essentially the same as that above. The control volumes for the velocities are of course displaced from those for ϕ. Interpolation is sometimes necessary to obtain convection velocities, densities, viscosities etc. at the required locations. In all cases our choice of interpolation practices is guided by the requirement that the resulting difference equation be conservative. If we denote the velocities in the x,y and z co-ordinate directions by u, v and w respectively, then the difference equations for momentum may be written:

$$C^{u}_P u_P = \sum_m C^{u}_m u_m + A_x (P_{x-} - P_P) + S^{u} \quad ; \tag{6}$$

$$C^{v}_P v_P = \sum_m C^{v}_m v_m + A_y (P_{y-} - P_P) + S^{v} \quad ; \tag{7}$$

$$C^{w}_P w_P = \sum_m C^{w}_m w_m + A_z (P_{z-} - P_P) + S^{w} \quad . \tag{8}$$

Here, the summations are over the six neighbouring velocities; and the coefficients in the equations are defined in an analogous fashion to those in (5). Finally, we complete the transformation to difference form by expressing the continuity relation (1) as:

$$[(\rho u)_{x+} - (\rho u)_P] A_x + [(\rho v)_{y+} - (\rho v)_P] A_y + [(\rho w)_{z+} - (\rho w)_P] A_z = 0. \tag{9}$$

2.3 The SIMPLE procedure. This 'Semi-Implicit Method for Pressure-Linked Equations' solves the set (6) to (9) by a cyclic series of guess-and-correct operations, wherein the velocities are first calculated by way of the momentum equations for a guessed pressure field, and then the latter, and later the velocities, are adjusted so as to satisfy continuity.

The first step in the cycle is straightforward: thus the guessed pressures (which may be initial guesses, or values from a previous cycle), denoted by P^*, are substituted into linearised† versions of (6) – (8). These are thus solved to yield a field of intermediate velocities u^*, v^* and w^* which will not, unless the solution has been reached, satisfy continuity.

It is here that the main novelties of the procedure enter, in the manner of satisfying the continuity requirement. The approach is to substitute for the velocities in eqn. (9) relations of the form:

$$u_P = u_P^* + A_P^u \, (P'_{x-} - P'_P) \quad ; \qquad (10)$$

$$v_P = v_P^* + A_P^v \, (P'_{y-} - P'_P) \quad ; \qquad (11)$$

$$w_P = w_P^* + A_P^w \, (P'_{z-} - P'_P) \quad ; \qquad (12)$$

where P' is a pressure correction, and the A's bear the following relation to coefficients in the momentum equations:

$$A_P^u \equiv A_x / C_P^u \; ; \qquad A_P^v \equiv A_y / C_P^v \; ; \qquad \text{and} \qquad A_P^w \equiv A_z / C_P^w \; .$$

The result is the finite-difference equivalent of a Poisson equation for P', viz:

$$C_P^P \, P'_P = \sum_n C_n^P \, P'_n + S^P \; . \qquad (13)$$

Here the summation sign has the usual meaning, and the coefficients are given by:

$$S^P \equiv [(\rho u^*)_{x+} - (\rho u^*)_P] \, A_x + [(\rho v^*)_{y+} - (\rho v^*)_P] \, A_y + [(\rho w^*)_{z+} - (\rho w^*)_P] \, A_z \; ;$$

$$C_P^P \equiv \sum_n C_n^P \; ; \qquad C_{x-}^P \equiv \rho_{x-} \, A_x A_P^u \; ; \qquad (14)$$

with similar definitions for the other terms. S^P, it should be noted, is nothing more than the local mass imbalance of the intermediate velocity field: so, when continuity is everywhere satisfied, the pressure correction goes to zero, as would be expected.

Once the P' field has been obtained from (13), it is a straightforward matter to update the pressures and velocities (from eqns. 10–12): then, if necessary, they may be used as guesses for a new cycle. If there are ϕ's to be calculated, they may be fitted in at a convenient stage in the cycle: often the choice is arbitrary.

Because the SIMPLE procedure computes the variable fields successively, rather than simultaneously, it is highly flexible in respect of the methods of solution which it will admit for the difference equations. For the present calculations, we have employed a line-iteration method, wherein the unknown variables along each grid line are calculated by application of the tridiagonal matrix algorithm, on the assumption that values on neighbouring lines are known. This operation is performed in turn on the sets of lines lying in the x, y and z directions: it usually suffices to perform one such 'triple sweep' on the velocities and ϕ's, and three sweeps on P', per cycle of calculation. This method is substantially faster than point iteration; however it must be stressed that when even more economical methods become available, they may readily be incorporated into the procedure.

<u>2.4 The SIVA procedure.</u> This procedure derives its name from the novel way in which it combines point iteration with SImultaneous Variable Adjustment. With this combination, it is possible to satisfy simultaneously, on a local basis, the equations

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† The coefficients and source terms are evaluated from the previous cycle, and held constant.

for momentum and continuity: although this balance is later destroyed when neighbouring nodes are visited, the net effect is to reduce the residual sources, and so procure convergence.

The procedure involves the adjustment, as each node is visited, of 7 variables, namely the pressure P, and the 6 surrounding velocity components, u_P, u_{x+}, v_P, v_{y+}, w_P and w_{z+}. The formulae for the variable adjustments are obtained by algebraic solution of: the continuity equation (9); and linearised versions of the momentum equations for the six velocities, expressed in the following form:

$$u_P = \alpha_P^u u_{x+} + \beta_P^u p_P + \gamma_P^u \quad ; \tag{15}$$

$$v_P = \alpha_P^v v_{y+} + \beta_P^v p_P + \gamma_P^v \quad ; \tag{16}$$

$$w_P = \alpha_P^w w_{z+} + \beta_P^w p_P + \gamma_P^w \quad ; \tag{17}$$

with similar expressions for u_{x+}, v_{y+} and w_{z+}. The quantities α, β and γ in these equations are readily deducible from the parent equations (6)-(9), whose terms involving variables outside of the 'SIVA cluster' have been swept into the γ's, and regarded (temporarily) as knowns. It is a straightforward matter to manipulate this set into equations which contain only the known coefficients on the right-hand sides: details will not be given here.

SIVA proceeds in all other respects in the manner of a normal point-iteration procedure: thus the grid is repeatedly swept, until the residual sources of the difference equations are reduced to acceptably small values. As with the SIMPLE method, the calculation of ϕ's is fitted in where appropriate.

3. APPLICATIONS

3.1 Test calculations. The SIMPLE and SIVA procedures were initially tested by application to a class of problems involving the laminar motion of a fluid in a cubic enclosure of side H, which has one wall moving at a steady velocity V in its own plane.

For a coarse mesh of 10 equally-spaced intervals, in each direction, convergent solutions were obtained for artificially high Reynolds numbers (based on V and H) in excess of 10^6. The SIMPLE procedure did exhibit some signs of instability in the initial stages of the calculations at the higher Reynolds numbers: this however could easily be cured by straightforward under-relaxation of P' (with a factor of about 0.7), often in the initial stages only. Although no other solutions were available for comparison, the predictions were entirely plausible, and two-dimensional versions of both methods agreed to within a few percent with Burggraf's (1966) fine-mesh computations. The initial studies confirmed that the two methods gave equal accuracy and numerical stability, but the SIMPLE method proved to be appreciably more economical of computing time than SIVA. It is therefore the former which we currently favour in our work.

In subsequent studies, SIMPLE has been successfully applied to several problems of practical interest, including the prediction of flow, heat transfer and chemical reaction in a three-dimensional furnace (Patankar and Spalding, 1972b) and the calculation of the steady-state and transient behaviour of a shell-and-tube heat exchanger (Patankar and Spalding, 1972c). Flows with strong effect of compressibility, and with distributed internal resistances, have also been predicted by the SIMPLE method.

3.2 The building problem. As a further example of the type of problem for which the SIMPLE method is well-suited, we here present calculations of the simulated (laminar) flow of wind past the slab-sided 'building', depicted in Fig. 3. The oncoming wind varies in strength in a parabolic fashion with distance from the ground, and is directed normal to the face of the building. An additional feature is a chimney located upwind of the building: the path of the effluent from this is also followed numerically.

The grid employed for the calculations had 10 nodes in each direction: non-uniform spacing was employed so as to cause the nodes to be concentrated near the building, and more widely-spaced elsewhere. The domain of solution, measured in building heights H, extended approximately ±5H in the mainstream (z) direction, and 8H in both the vertical (y) and lateral (x) directions. The plane x=0 was prescribed as a plane of symmetry, while at all other free boundaries the flow was presumed to be undisturbed by the presence of the building. The Reynolds number, based on H and the undisturbed velocity w_H at y=H, was approximately 100, in this purely illustrative example.

The results are displayed in Fig. 4, in the form of plots, at a number of constant-z planes, of: contours of constant mainstream velocity; vectors representing the direction and magnitude of the resultant velocities in the xy planes; isobars; and contours of the effluent concentration.

Taken together, the velocity and pressure plots reveal a consistent and plausible pattern of behaviour: thus the build-up of pressure in front of the building provokes reverse flow (indicated by the negative-w contour) in the low-velocity region near the ground, and deflects the wind away from the building. Downstream, the low-pressure zone behind the building also gives rise to reverse and lateral flows: now however the fluid is drawn inwards.

The concentration contours show that the effluent plume initially spreads downwards, thereby causing relatively high concentrations at the upwind face of the building. The flow around the latter then deflects the plume upwards, so that the concentration on the downwind face is lower, although still appreciable.

Although it cannot be claimed that a laminar-flow calculation on a relatively sparse grid is quantitatively representative of the real situation, the above results are probably at least qualitatively correct: moreover, they were obtained at a quite modest cost (approximately 100 seconds on a CDC 6600 machine).

4. DISCUSSION AND CONCLUSIONS

4.1 Assessment of the procedures. Experience with the SIVA and SIMPLE algorithms, which have now been applied to a large number of flow situations of varied type, has demonstrated the great flexibility and stability that results from using implicit finite-difference formulations, with the hybrid difference scheme. It has also shown that the line-by-line nature of the SIMPLE adjustment procedure makes for greater economy of computer time than the point-by-point SIVA adjustment. The slightly-reduced stability of SIMPLE can be rectified by an inexpensive under-relaxation. The authors therefore intend to concentrate on SIMPLE in their future work.

4.2 Prospects for future development. The example of Fig. 3 shows that the calculation procedure can be employed for predicting practically-important phenomena which, at present, can be predicted only by way of rather expensive and time-consuming experiments. However, a consideration of the shortcomings of that example shows also much development still to be done. First of all, the calculation was performed for a low-Reynolds-number laminar flow; but flows over real buildings are of high Reynolds number, and turbulent. It is therefore necessary to incorporate into the calculation procedure "turbulence models", of the kind recently surveyed by Launder and Spalding (1972).

Secondly, it will have been observed that the calculation task was made especially easy by the fact that four of the boundaries of the domain of integration were treated as impervious to matter, while the inlet boundary was as one at which the velocity distribution was known. In reality, the elliptic nature of the flow ensures that the presence of the building modifies the velocity distribution at these boundaries: some economical means of calculating this modification needs to be built into the calculation procedure.

Finally, buildings are not simply rectangular blocks; sometimes the departures from simplicity of form may have significant aerodynamic effects. It is therefore necessary to arrange that significant minor details of the surface, for example its distribution of roughness, can be allowed with the calculation scheme, without necessitating excessive refinement of the grid.

If these problems can be speedily surmounted, there is every reason to expect that numerical computations will replace model experiments for civil-engineering aerodynamics, furnace design, and many areas of hydraulic and aeronautical engineering. No difficulties of principle appear to stand in the way of these developments, and none of the difficulties of detail is of a kind which has not been surmounted elsewhere.

5. REFERENCES

Amsden A.A. and Harlow F.H. (1970). "The SMAC Method". Los Alamos Scientific Laboratory Report No. LA-4370.

Caretto L.S., Curr R.M. and Spalding D.B. (1972). "Two numerical methods for three-dimensional boundary layers". Comp. Methods in Appl. Mech. and Eng., 1, pp.39-57.

Chorin A.J. (1968) "Numerical solution of the Navier-Stokes equations". Maths. of Computation, 22, No.104, pp. 745-762.

Gosman A.D., Pun W.M., Runchal A.K., Spalding D.B. and Wolfshtein M. (1969). Heat and Mass Transfer in Recirculating Flows. Academic Press, London.

Harlow F.H. and Welch J.E. (1965). "Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface". Physics of Fluids, 8, No. 12, pp. 2182-2189.

Launder B.E. and Spalding D.B. (1971). "Turbulence models and their application to the prediction of internal flows". Proc. I. Mech. Eng. Symposium on Internal Flows, Salford.

Patankar S.V. and Spalding D.B. (1970). "Heat and Mass Transfer in Boundary Layers". Intertext Books, London, Second Ed.

Patankar S.V. and Spalding D.B. (1972a). "A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows". To be published in Int. J. Heat and Mass Transfer.

Patankar S.V. and Spalding D.B. (1972b). "A computer model for three-dimensional flow in furnaces". To be presented at 14th Symposium on Combustion.

Patankar S.V. and Spalding D.B. (1972c). "A calculation procedure for steady and transient performance of a shell-and-tube heat exchanger". To be presented at the Int. Summer School at Trogir, Yugoslavia.

Pracht W.E. (1970). "A numerical method for calculating transient creep flows". Los Alamos Scientific Lab. Rept. LA-DC-11312.

Runchal A.K. (1970). "Convergence and accuracy of three finite-difference schemes for a two-dimensional conduction and convection problem". Imperial College Mech. Eng. Dept. Rept. EF/TN/A/24.

Spalding D.B. (1972a). "Mathematical models of continuous combustion". Proc. Gen. Motors Symposium on Emissions from Continuous Combustion Systems, Detroit.

Spalding D.B. (1972b). "A novel finite-difference formulation for differential expressions involving both first and second derivatives". To be published in Int. J. Num. Methods in Eng., 4.

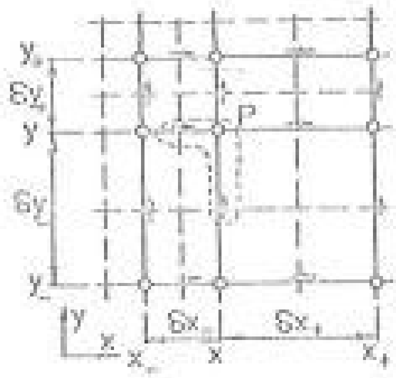

Fig. 1. The Staggered Grid

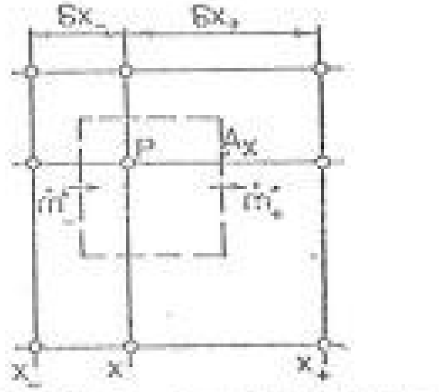

Fig. 2. Notation for x-direction fluxes

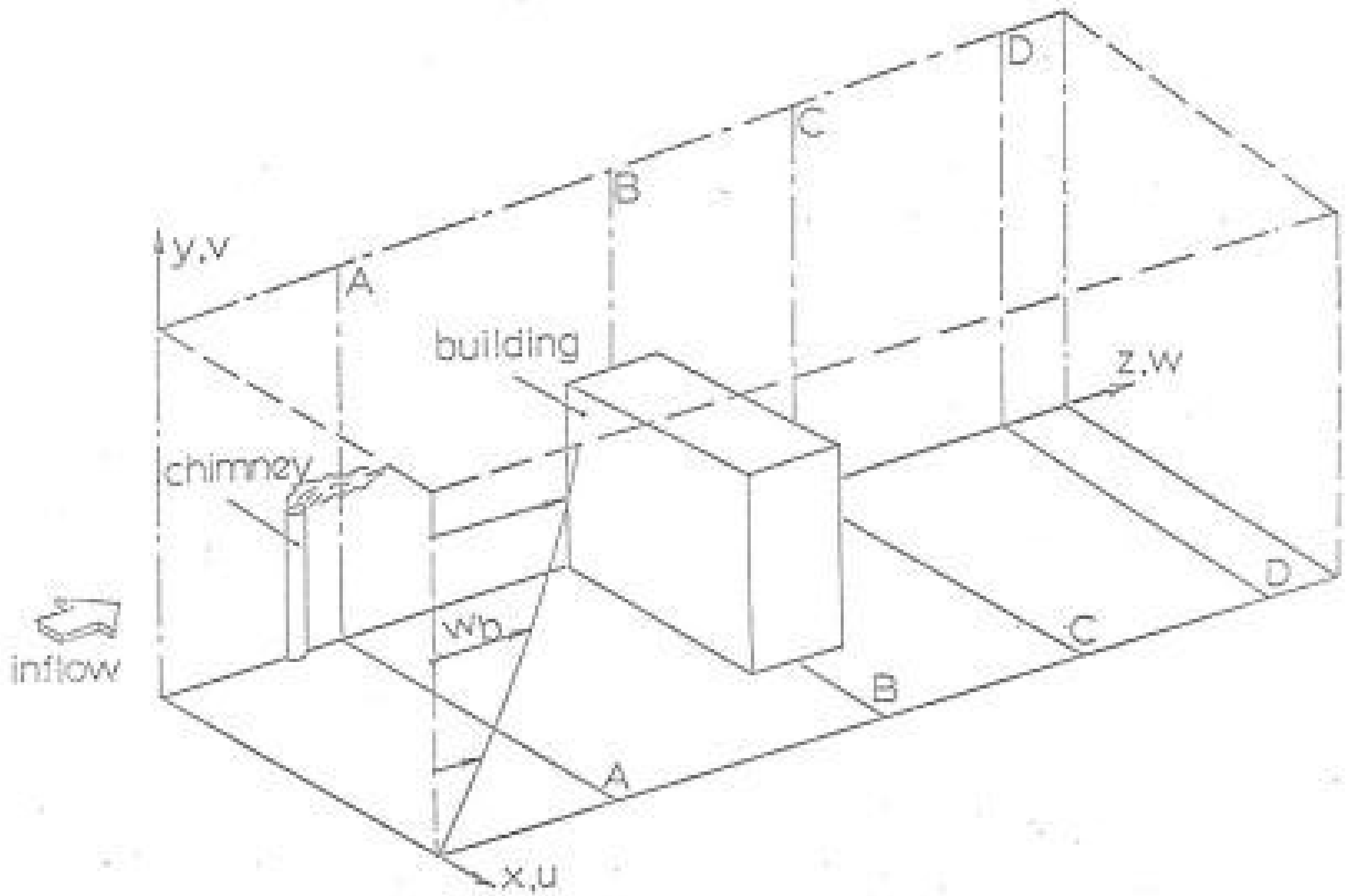

Fig. 3. Illustration of the flow-past-building problem

Fig. 4. Results of the flow-past-building problem. (a) main-flow velocity contours (w/w_{max}); (b) velocity vectors in cross-stream planes; (c) static-pressure contours $[2(P-P_{ref})/\rho w_g^2]$; (d) effluent concentration contours (source concentration = 100)

APPENDIX 'E'

Calculation of the Three-Dimensional Boundary Layer with Solution of all Three Momentum Equations.

by

S V Patankar, D Rafiinejad and D B Spalding

Computer Methods in Applied Mechanics and Engineering 6, pp 283-292, 1975

COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 6 (1975) 283–292

CALCULATION OF THE THREE-DIMENSIONAL BOUNDARY LAYER WITH SOLUTION OF ALL THREE MOMENTUM EQUATIONS

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The paper describes the application of a calculation procedure to three flow situations which can be characterised as three-dimensional boundary layers. Unlike most of the published methods, the present procedure solves all momentum equations and takes full account of the pressure variation in directions normal to the main-flow direction. The applications demonstrate that, when the boundary conditions exhibit certain discontinuities, only the solution of all three momentum equations can give satisfactory accuracy. The results of the present calculations are compared with available similarity solutions wherever possible.

Nomenclature

x	distance in main-flow direction
y, z	cross-stream coordinates
ρ	density
ν	kinematic viscosity
u, v, w	velocity components in x, y, z directions, respectively
$\bar{p}$	pressure in the main-direction momentum equation
p	pressure in the cross-stream momentum equations
T	temperature
η, ζ	similarity variables: $\eta \equiv y\,(u_\infty/\nu x)^{1/2}$, $\zeta \equiv z\,(u_\infty/\nu x)^{1/2}$
a, b, c	constants
δ	boundary-layer thickness

subscripts	*pertaining to*
∞	free-stream condition
w	wall condition

1. Introduction

There are several techniques available for solving three-dimensional boundary-layer flows, but most of the published procedures solve only two momentum equations and obtain the third component of velocity from the continuity equation. Such methods are discussed in a review [1] and include those in [2]–[6]. There are, however, some phenomena which require all three momentum equations to be taken into account. The authors know of only four finite difference methods which can solve the problem of the latter kind: Caretto, Curr and Splading [7] have developed two procedures, and more recently Patankar and Spalding [8] and Briley [9] have developed techniques to solve the three-dimensional boundary-layer equations in general form. A.J. Baker presents an interesting finite element method in [10].

The present paper applies the methods of [8] to three particular problems. The first problem is such that our general method would reduce identically to the conventional method. For the second problem, both kinds of method are applicable, but they yield results which are different. The third problem cannot be solved by the conventional methods and a general method of the type in [8] is required.

Attention is confined here to uniform-property laminar flows. The results of our calculations are compared with available solutions.

2. Outline of the prediction procedure

The prediction procedure of [8] will now be briefly outlined.

2.1. The differential equations

The problem is that of solving the following set of parabolic partial differential equations with appropriate boundary conditions:

Continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,, \tag{2.1}$$

Momentum

$$u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+w\,\frac{\partial u}{\partial z}=-\frac{1}{\rho}\,\frac{\partial \bar{p}}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right), \tag{2.2}$$

$$u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+w\,\frac{\partial v}{\partial z}=-\frac{1}{\rho}\,\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right), \tag{2.3}$$

$$u\,\frac{\partial w}{\partial x}+v\,\frac{\partial w}{\partial y}+w\,\frac{\partial w}{\partial z}=-\frac{1}{\rho}\,\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right), \tag{2.4}$$

Thermal Energy

$$u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}+w\,\frac{\partial T}{\partial z}=\frac{\nu}{\mathrm{Pr}}\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right). \tag{2.5}$$

These equations are valid for three-dimensional, steady, laminar, uniform-property, boundary-layer flow. The dependent variables u, v, w, p, and T are determined as functions of the Cartesian coordinates x, y, and z. (Although the method of [8] can handle non-uniform properties, that facility is not needed for the present paper.)

In the above set of equations, viscous and heat-conductive actions across planes of constant x are neglected so that no influences from downstream can penetrate upstream. This neglect is consistent with the nature of the flows under consideration and contributes to the parabolic character of the equations. A further contribution is the different treatment of the pressure in the x-momentum equation from that appropriate to the cross-stream momentum equations. These two pressures are shown by different symbols $\bar{p}$ and p. The gradient $\partial\bar{p}/\partial x$ is assumed to be a known function of position, deducible from conditions outside the boundary layer. Since the equations are parabolic, integration can proceed by "marching" in the x direction.

2.2. The finite difference equations

A system of rectangular grids is chosen for the $y-z$ planes. The size of the rectangle is allowed to change as we move in the x direction. The rate of change of this is so chosen that our grid just encloses the flow region of interest.

The field variables are stored in a staggered fashion: u, p and T are stored for main grid nodes, while v and w, respectively, are stored for the center points of the y-wise and z-wise links joining these nodes.

The differential equations are written in finite difference form by integration over a control volume surrounding each node or center point. The equations are solved by the line-by-line application of the tridiagonal matrix algorithm.

2.3. The solution procedure

Solution of the finite-difference equations proceeds as follows:

a. The pressure distribution in the $y-z$ plane is guessed (upstream values are good guesses).
b. A first approximation to the cross-stream velocity field is then obtained from a solution of the momentum equations for v and w.
c. The longitudinal pressure gradient $\partial\bar{p}/\partial x$ is known from the conditions outside the boundary layer. The x-component of velocity, u, can therefore be obtained directly from the relevant finite difference equations.
d. The velocities obtained in steps b and c do not in general satisfy continuity, because only a guessed pressure distribution has been used. Therefore, corrections to this pressure field are calculated such that corresponding corrections to v and w will bring the velocity field in conformity with the continuity equation.
e. The thermal energy equation is then solved to yield the temperature field for the plane.

Steps a to e complete the set of computations at a given forward position x; they are repeated plane-by-plane until the whole field has been swept from upstream to downstream.

2.4. A simplified version of the method

The method just outlined takes full account of the diffusion fluxes and pressure gradients in both cross-stream directions; it is this feature which distinguishes the method from those used in most three-dimensional boundary-layer investigations, in which the diffusion fluxes are assumed to be important in only one of the cross-stream directions and the pressure gradient only in the other.

One of the aims of the present paper is to establish whether these approximations can lead to appreciable error in the problems considered here. For this purpose, rather than to program a published conventional procedure, we have found it conveient to cast the general method of [8] into a simplified form.

In this simplified version both diffusion fluxes are retained, but $\partial p/\partial y$ is put equal to zero, so that the whole pressure field is specified and no guess-and-correct procedure is needed. Then u and w can be obtained directly from the relevant momentum equations, and v follows from the continuity equation.

Henceforth, the original method of [8] will be referred to as method I and this simplified version as method II.

3. Laminar boundary layer with pressure gradient and injection

Laminar flow over a plane surface is considered, with and without injection at the wall.

3.1. Flow without injection

Statement of the problem: We consider a flow over a flat surface with the free-stream conditions: $u_\infty = a$, $w_\infty = bx$, where a and b are constants. Yohner and Hansen [11] have obtained similarity solutions for this case, and Krause et al. [12] have applied their finite difference technique to this problem.

The computations are carried out for $a = b = 1$ in the region ABCD of the y–z plane at each forward location x (fig. 1). Due to the nature of the free-stream conditions, the similar solution of this problem is independent of z. Therefore, the distance AD is kept fixed, independent of x and much larger than AB. However, AB expands downstream in proportion to $(\nu x/u_\infty)^{1/2}$. As a result, an expanding grid was used in the y-direction such that $\text{AB}/x = \text{const}\,(xu_\infty/\nu)^{-1/2}$. All runs were performed with a uniform, 11 × 11 grid, which was found to give sufficient accuracy.

Both methods I and II were used for this problem. When method II was used, pressure gradients $\partial p/\partial z = -b\rho u_\infty$ and $\partial p/\partial y = 0$ were specified.

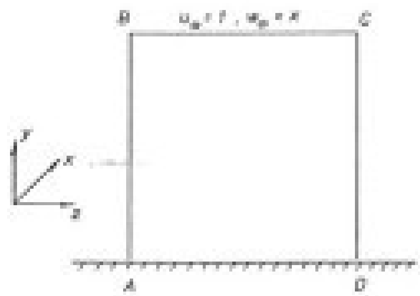

Fig. 1. Coordinates and calculation region for problem of section 3.1.

Results

The velocity profiles are plotted in figs. 2a and 2b, where comparisons are made with the results of Yohner and Hansen [11]. Since u_∞ is a constant and w_∞ is a function of x alone, the u- and v-distributions must be identical with the solution of the Blasius equation, while the w-distribution has a maximum within the boundary layer due to the curvature of the external streamlines. The

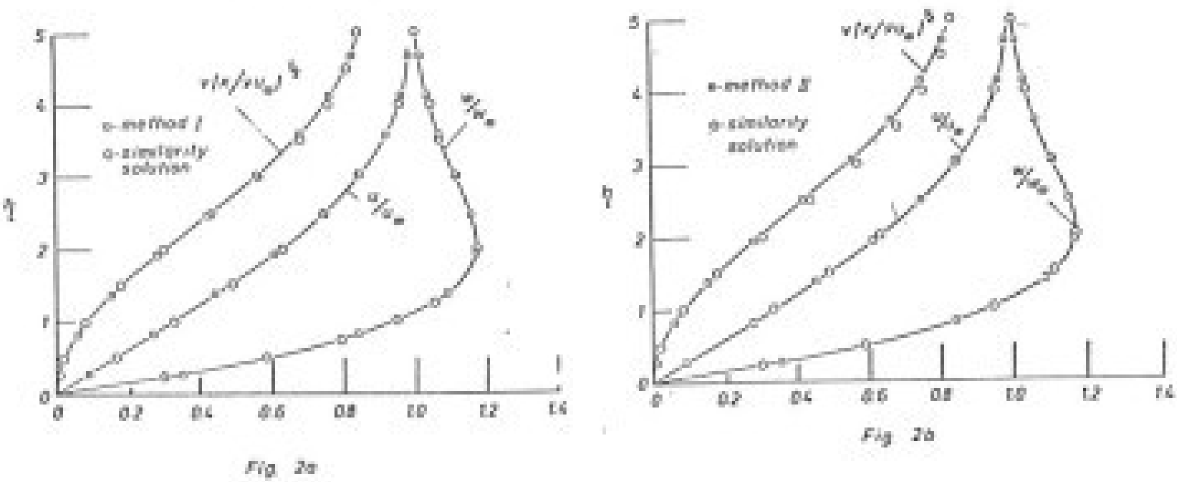

Fig. 2. Similarity solution of laminar three-dimensional flow over an impervious plane wall, $u_\infty = 1.0$, $w_\infty = x$.

agreement of our numerical solution with those of [11] is seen to be good. Due to the nature of the flow, no difference is found between methods I and II, and both methods are equally valid in this case. However, the computer time is somewhat larger (about 30%) for method I than for method II.

3.2. Flow with injection

Statement of the problem: Laminar flow with the same free-stream conditions ($u_\infty = a$ and $w_\infty = bx$) is now considered over a partially porous plane wall. The porous region extends along a strip in the x-direction; fluid is blown through it into the boundary layer.

The calculation region in the $y-z$ plane at a particular x is shown in fig. 3, where the wall AD is porous along the region EF. The blowing velocity at the wall is $v_w = \text{const}(\nu u_\infty/x)^{1/2}$. The distance AD is kept fixed but AB expands downstream according to $AB/x = \text{const}(xu_\infty/\nu)^{-1/2}$, wherein the constant is selected by numerical experiment so that no change is observed in the velocity profiles as the constant is varied. Computations were performed with both methods I and II with an 11×15 grid; this gave sufficient accuracy.

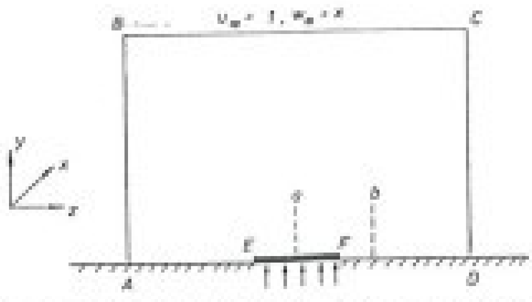

Fig. 3. Flow geometry and calculation region for problem of section 3.2.

Results

The results of calculations for two different blowing rates corresponding to $v_w(x/\nu u_\infty)^{1/2} = 0.25$ and 0.5 are presented here. The results of methods I and II are compared at two different spanwise locations:

(a) over the blowing region,

(b) outside but close to the blowing region (fig. 3).

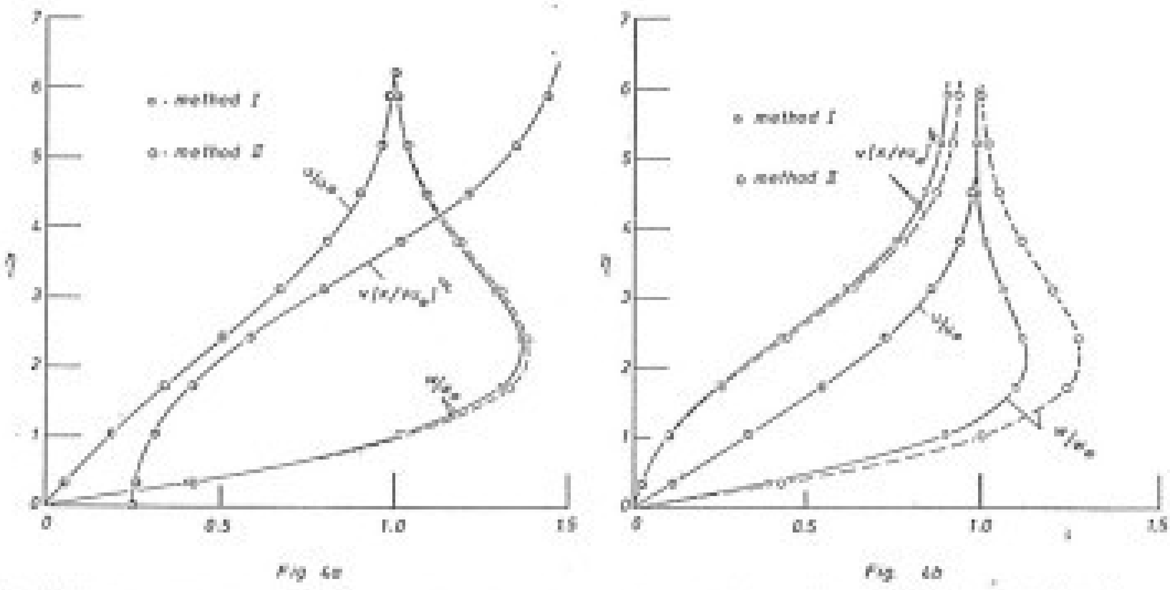

Fig. 4. Velocity profiles of laminar three-dimensional flow over a partially porous plane wall, $u_\infty = 1.0$, $w_\infty = x$, $v_w(x/\nu u_\infty)^{1/2} = 0.25$, $x = 0.718$.

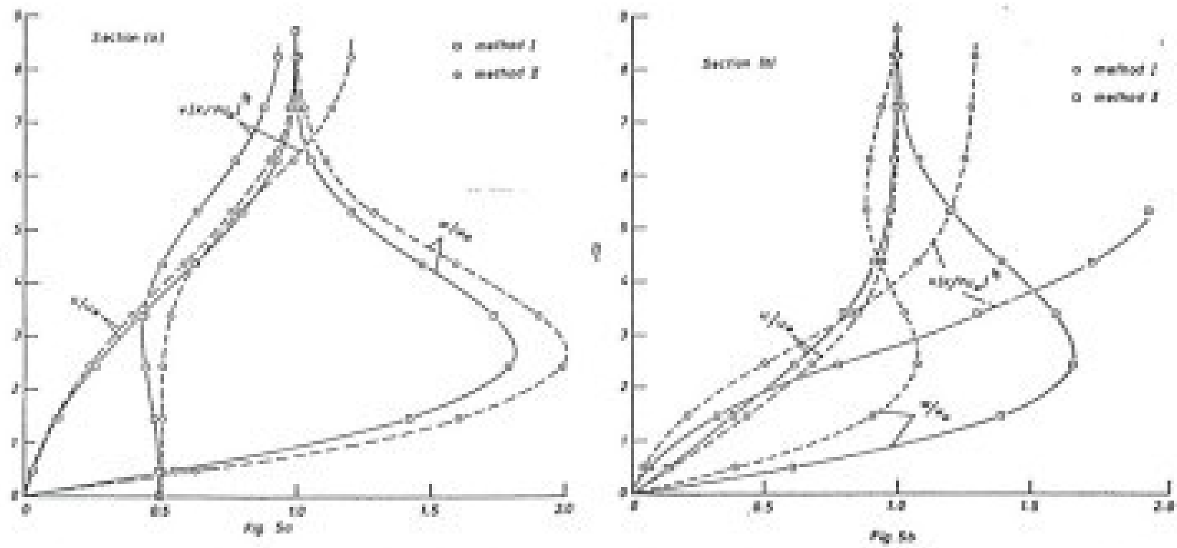

Fig. 5. Velocity profiles for laminar three-dimensional flow over a partially porous plane wall, $u_\infty = 1.0$, $w_\infty = x$, $v_w(x/\nu u_\infty)^{1/2} = 0.5$, $x = 1.38$.

Figs. 4a and 4b compare the two methods for $v_w(x/\nu u_\infty)^{1/2} = 0.25$, and figs. 5a and 5b correspond to the value of 0.5. It is noted that the two methods produce different results for this problem, and the differences are more important over section (b). Particularly for the larger blowing rate, large differences are noticed in the velocity profiles.

It is of course the method I results which must be regarded as the more accurate. The terms in the equations which method II neglects are evidently not negligible in this case.

4. Laminar flow along a rectangular corner

4.1. Statement of the problem

Laminar flow along a corner formed by the intersection of two perpendicular flat plates is considered (fig. 6). The free-stream velocity is assumed to be of the form $u_\infty = cx^m$, where c and m are constants and x is the distance along the corner from the leading edge. The computations at each forward location x are performed in a rectangular region ABCD whose boundaries are effectively two dimensional. A uniformly spaced grid is used in the y–z plane which expands as the calculations proceed downstream. The rates of growth of AD and AB with x are assumed equal and proportional to the rates of growth of the two-dimensional boundary layer far from the corner region, i.e. $\mathrm{d(AB)}/\mathrm{d}x = C_1\,\mathrm{d}\delta(x)/\mathrm{d}x$, where $\delta(x)$ is the thickness of the corresponding two-dimensional boundary layer, and C_1 is a constant.

Boundary conditions: "No-slip" conditions are used along the AB and AD boundaries. Along the CD and BC boundaries, which are far from the corner, two-dimensional velocity distributions are specified according to the integral solution of two-dimensional flow over a flat plate with a given pressure gradient [13]. The latter boundary conditions are not essential, since outer-edge conditions have no influence on the central part of the solution if the integration domain is sufficiently large.

Heat transfer calculations are performed with wall temperatures T_w specified according to the following power law; $T_w - T_\infty = \mathrm{const}\, x^{2m}$, where T_∞ is the free-stream temperature. This wall-temperature distribution gives a self-similar temperature field in the corner [14].

A uniform 15 × 15 rectangular grid is used in all runs. Method II is not valid in this case and hence only method I has been used.

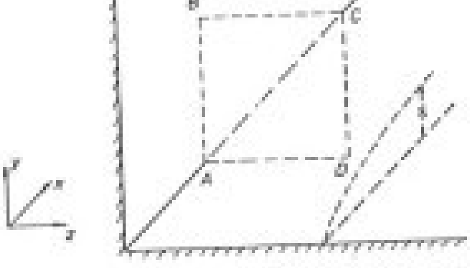

Fig. 6. Coordinates and geometry of flow along a rectangular corner.

4.2. Results

Two sets of calculations have been performed: (1) with uniform free-stream velocity $m = 0$; and (2) with favorable pressure gradient $m = 0.5$.

Zamir and Young [15] have made an extensive experimental investigation of the flow in a corner. They have found that the laminar flow in the corner breaks down at relatively small streamwise Reynolds numbers. Therefore, their data are either in the transition or fully turbulent regions and are not comparable with the present results. A number of investigators, however, have obtained similarity solution to the problem by integral techniques. Schlichting [14] reports the results of the work by R. Vasanta. The present results are compared with the latter work.

Velocity distributions

Constant-velocity contours of the similar solutions are shown on figs. 7a and 7b for $m = 0$ and 0.5, and compared with the integral solutions of Vasanta. The agreement is very good except for slight discrepancies near the free-stream region. Perhaps the latter are caused by the coarseness of the grid (15 × 15); but, of course, the integral solutions are themselves inexact.

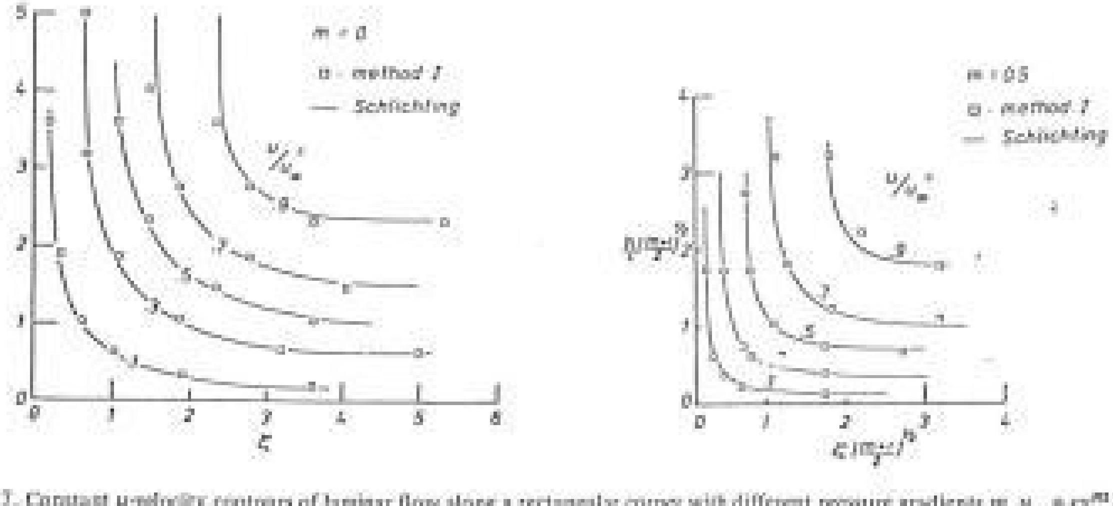

Fig. 7. Constant u-velocity contours of laminar flow along a rectangular corner with different pressure gradients m. $u_\infty = cx^m$, $c = 100$.

Temperature distributions

The constant-temperature contours are also plotted for $m = 0$ and 0.5 on figs. 8a and 8b. The Prandtl number Pr equals 0.7, frictional heating is neglected, and fluid properties are assumed constant. The temperature profiles are close to their corresponding u-distributions, as expected, since the Prandt number is close to unity.

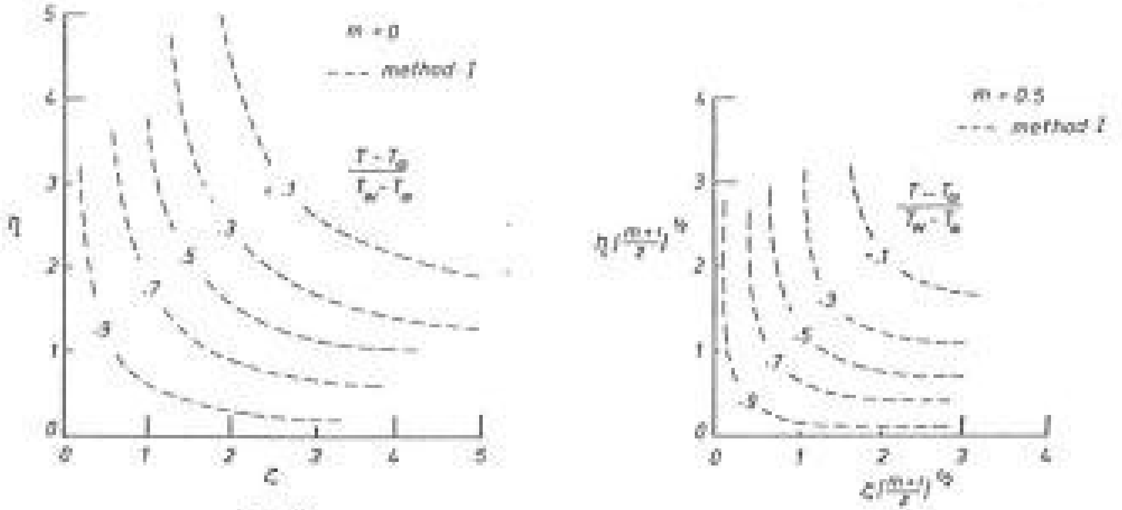

Fig. 8a Fig. 8b

Fig. 8. Temperature distribution in laminar flow along a rectangular corner with different pressure gradients m. $u_\infty = cx^m$, $T_w - T_\infty = c'x^{2m}$, $c = 100$, $c' = 100$.

4.3. Effect of grid-expansion rate

As mentioned earlier, along with the marching process the grids in the y–z plane were expanded at the rate $c_1 \, d\delta(x)/dx$. The influence of the constant c_1 on the similarity solutions was studied. Fig. 9 shows the effect of C_1 on the u-velocity distribution along the bisector of the corner. The results suggest that $c_1 = 1.5$ is the optimum value; it allows the solution to be independent of C_1 without having a large number of grid points in the free-stream region.

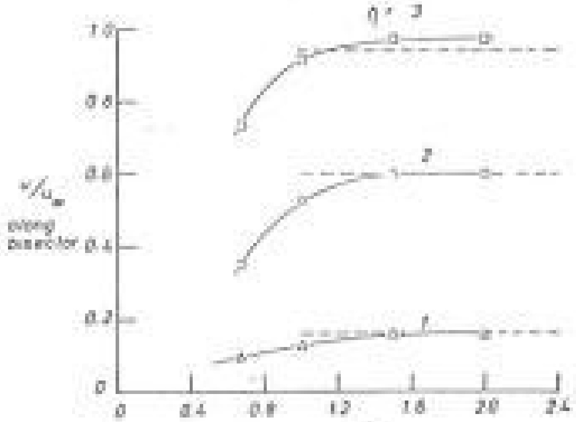

Fig. 9. Effect of grid expansion rate on solution of laminar flow along a rectangular corner.

5. Conclusions

The numerical procedure of [8] has been applied to the prediction of certain three-dimensional external boundary-layer flows. This procedure is applicable in cases where diffusion fluxes and pressure gradients in one or both of the cross-stream directions are significant as in the cases of a partially porous wall and flow in a corner. The method, of course, also produces satisfactory results for problems for which the more restrictive existing procedures are valid. For these, however, a special version (method II) can be employed which uses about three-quarters of the computer time of method I.

Acknowledgement

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References

[1] S.V. Patankar, On available calculation procedures for steady, three-dimensional boundary layers, Report No. BL/TN/A/44 (Mech. Eng. Dept., Imperial College, London, Apr. 1971).
[2] M.G. Hall, A numerical method for calculating steady three-dimensional laminar boundary layers, RAE Tech. Rep. 67145 (Jan. 1967).
[3] H.A. Dwyer. Solution of a three-dimensional boundary layer flow with separation, AIAA J. 6 (1968) 1336–0000.
[4] T.K. Fannelop, A method of solving the three-dimensional laminar boundary-layer equations with application to a lifting re-entry body, AIAA J. 6 (1968) 1075–0000.
[5] J.F. Nash, The calculation of three-dimensional turbulent boundary layers in compressible flow, J. Fluid Mech. 37 (1969) 625–000.
[6] E. Krause and E.H. Hirschel, Exact numerical solutions for three-dimensional boundary layers, Second International Conference on Numerical Methods in Fluid Dynamics (Univ. California, Berkeley, Sept. 1970).
[7] L.S. Caretto, R.M. Curr and D.B. Spalding, Two numerical methods for three-dimensional boundary layers, Comp. Meths. Appl. Mech. Eng. 1 (1972) 39–00.
[8] S.V. Patankar and D.B. Spalding, A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows, IJHMT 15 (1972) 1787–0000.
[9] W.R. Briley, A numerical method for predicting three-dimensional viscous flows in ducts, Report No. L110888 – 1 (United Aircraft Labs., Nov. 1972).
[10] A.J. Baker, Finite element solution theory for three-dimensional boundary layer flows, Comp. Meths. Appl. Mech. Eng. 4 (1974) 367–386.
[11] P.L. Yohner and A.G. Hansen, Some numerical solutions of similarity equations for three-dimensional laminar incompressible boundary layer flows, TN 4370 (NACA, 1958).
[12] E. Krause, E.H. Hirschel and Th. Bothmann, Normal injection in a three-dimensional laminar boundary layer, AIAA J. 7 (1969) 1969–0000.
[13] H. Schlichting, Boundary layer theory, 4th ed. (McGraw-Hill, 0000, ch. 12).
[14] H. Schlichting, A survey of some recent research investigations on boundary layers and heat transfer, J. Appl. Mech. (19?1) 289–000.
[15] M. Zamir and A.D. Young, Experimental investigation of the boundary layer in a streamwise corner, Aero. Quart. (1970) 313–000.

APPENDIX 'F'

Fluid Flow and Heat Transfer in Three-Dimensional Duct Flows.

by

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FLUID FLOW AND HEAT TRANSFER IN THREE-DIMENSIONAL DUCT FLOWS

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Abstract—A calculation procedure is described for three-dimensional duct-flow situations which are partially-parabolic in nature, i.e. those in which convective influences pass only downstream, diffusive influences are directed across the stream, but influences are transmitted from downstream regions to upstream ones by way of pressure. The numerical calculation procedure handles such flows economically; it stores the pressure as a three-dimensional array, but other variables two-dimensionally. As an illustration, the results from an application of the calculation procedure are compared with those of a parabolic calculation procedure.

NOMENCLATURE

A, B, C, D, coefficients in the finite-difference equations;
J, diffusion flux;
K, constant;
p, pressure;
S, source or sink term;
u, velocity along x-direction;
v, velocity along y-direction;
w, velocity along z-direction;
x, y, z, coordinate directions.

Greek symbols

ρ, density;
τ, shear stress;
ϕ, general variable.

Subscripts

D, E, W, N, S, U, P, e, w, n, s, p, refer to grid and interface locations;
x, y, z, coordinate directions;
u, v, w, refer to corresponding velocities.

1. INTRODUCTION

1.1. *Classification of steady-flow situations*

IT HAS been useful in numerical fluid mechanics to divide steady-flow problems into two classes: elliptic and parabolic. Strictly speaking all flows except wholly supersonic ones are elliptic; this means that perturbations of conditions at any point of the flow can influence conditions at any other point. The mechanisms of these interactions are usually:

(i) Convection (i.e. downstream transmission along stream lines);
(ii) Conduction, diffusion and viscous action (i.e. dissemination in all directions by molecular intermixing);
(iii) Pressure transmission (e.g. the tendency of a fluid in a subsonic flow to move out of the way of a downstream obstacle before reaching it).

In "parabolic" flows, mechanisms (ii) and (iii) are weak enough to be ignored; and the flow configuration is free from "recirculation", so that mechanism (i) transmits effects only in one direction. Many boundary-layer, duct-flow and jet phenomena are of this parabolic kind; for often the Reynolds number is high enough to render the molecular actions insignificant *in* the streamwise direction; and the boundaries of the flow domain provoke no sharp curvatures of streamlines.

In the present paper however, attention is focussed upon a class of flow situations which is intermediate to the parabolic and elliptic categories. Such flows, here called "partially-parabolic", are characterised by:

(a) Absence of recirculation, so that mechanism (i) (convection) operates only in a single (downstream) direction;
(b) High Reynolds number, so that mechanism (ii) (molecular action) is significant only normal to the stream-lines;
(c) *Significant* curvature of boundaries, rendering (iii) (pressure transmission) the dominant transmitter of influences in an upstream direction.

1.2. *Examples of partially-parabolic flows*

Phenomena falling into the partially-parabolic class include:

(a) Flow in strongly-curved ducts, for example pipe bends in heat exchangers;
(b) Flow in turbine and compressor cascades;
(c) Flows in and near partially permeable resistances such as gauzes and screens of tubes or rods, as in the shells of some steam generators;
(d) Flows of lubricants in two-dimensional oil films.

1.3. *Significance for numerical computation*

Elliptic flows require computer storage of dimensionality equal to that of the flow; the storage dimensionality of a parabolic flow, by contrast, is one less than that of the flow. Consequently, since influences spread only in the downstream direction in parabolic flows, *marching integration* can be employed; and there is no need to retain in store flow variables for more than the immediately-upstream plane or line. For elliptic phenomena by contrast, it is necessary to retain all upstream values in store; for they may have to be altered again in the light of adjustments still to be made downstream; an iterative procedure is thus always required.

For *partially-parabolic* flows, the requirements are intermediate: only the pressure requires to have storage dimensionality equal to the flow dimensionality; the other variables (i.e. velocity components, temperature, concentrations, etc.) require only the reduced dimensionality of parabolic flows. Thus the main advantage of a partially-parabolic situation, over the elliptic one, comes from the significant reduction in the storage requirement. This advantage is greatest for three-dimensional flows, as can be seen in the following calculations:

Suppose that 20 grid points are required in every direction for adequate coverage of the domain, i.e. 400 for a two-dimensional problem and 8000 for a three-dimensional one.

Suppose also that we are concerned, as is often the case, with three velocity components, pressure, temperature, two turbulence quantities and concentration, i.e. eight variables in all. If the flow is two-dimensional and elliptic, we need 8×400, i.e. 3200 storage locations; however, if it is partially-parabolic the storage is reduced to 400 (for pressure) $+ 7 \times 20$, i.e. to 540 locations, a reduction of 2660.

A three-dimensional elliptic problem with this grid fineness requires 64 000 storage locations; if however the process reduces to partially-parabolic form, the storage requirement is only $8000 + 7 \times 400$, i.e. 10 800, a reduction of 53 200. Such a reduction is of great value.

This being the case, it is perhaps remarkable that the partially-parabolic flow class seems to have escaped attention until now. Certainly there is every reason to recommend that wherever possible, three-dimensional flows should be treated as partially-parabolic instead of fully elliptic.

1.4. *Outline of the present contribution*

Calculation procedures for three-dimensional parabolic [2] and elliptic [1] flows have been available for some time; and they have been applied to various flow configurations. In this report we describe a numerical procedure for the calculation of *partially-parabolic flow situations*. Like the parabolic calculation procedure [2], the present procedure is of a finite-difference type and makes use of the SIMPLE* algorithm; but its distinctive features are:

(a) The pressure field alone is stored in a three-dimensional array, to be used in common for all the three momentum equations.

(b) An iterative, marching-integration procedure is adopted, whereby several sweeps of the flow domain are made; each sweep uses a better estimate of the pressure field, deduced from the observation of errors during the previous sweep. All other variables, e.g. velocities etc., are stored in two-dimensional arrays.

In Sections 2 and 3 of the report, the differential equations and the calculation procedure are explained; an illustrative example of partially-parabolic flow situation is described in Section 4, along with the application of the present procedure for its calculation. From comparisons made between the results using the parabolic and partially-parabolic procedures, it is observed that the partially-parabolic calculations display the expected flow-pattern and differ significantly from those obtained by using the parabolic procedure.

2. DIFFERENTIAL EQUATIONS SOLVED

The equations governing a partially-parabolic flow are the familiar Navier–Stokes equations for a steady flow but with diffusion in the predominant flow direction (z) neglected. In the (x, y, z) coordinate system, they are:

Mass conservation:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (2.1)$$

x-direction momentum:

$$\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) + \frac{\partial}{\partial z}(\rho w u) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + S_u \qquad (2.2)$$

y-direction momentum:

$$\frac{\partial}{\partial x}(\rho u v) + \frac{\partial}{\partial y}(\rho v v) + \frac{\partial}{\partial z}(\rho w v) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + S_v \qquad (2.3)$$

z-direction momentum:

$$\frac{\partial}{\partial x}(\rho u w) + \frac{\partial}{\partial y}(\rho v w) + \frac{\partial}{\partial z}(\rho w w) = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + S_w \qquad (2.4)$$

Transport of a scalar property, ϕ:

$$\frac{\partial}{\partial x}(\rho u \phi) + \frac{\partial}{\partial y}(\rho v \phi) + \frac{\partial}{\partial z}(\rho w \phi) = \frac{\partial J_\phi}{\partial x} + \frac{\partial J_\phi}{\partial y} + S_\phi \qquad (2.5)$$

In the above equations, u, v and w denote the velocities along the x, y and z directions; ρ represents the fluid density, and p the pressure. The "τ"s represent the shear stresses in the fluid; and J_ϕ stands for the flux of the property ϕ. The terms S_u, S_v, S_w and S_ϕ represent additional sources or sinks.

The differences between the above equations and those of elliptic and parabolic flows are the following:

(i) For an elliptic flow, the governing equations will contain also the shear stresses in the z direction; i.e.

*SIMPLE stands for semi-implicit method for pressure-linked equations.

terms such as $\partial\tau_{xz}/\partial z$, $\partial\tau_{yz}/\partial z$ etc. will appear in the corresponding equations.

(ii) For a parabolic flow, on the other hand, not only do the equations not contain the diffusion fluxes in the z direction but separate pressure fields have to be employed for the lateral and longitudinal momentum equations. The latter practice in parabolic flows is necessary to ensure that the pressure transmission of downstream events is negligible.

3. DETAILS OF THE SOLUTION PROCEDURE

The above-described differential equations for a partially-parabolic flow are solved using a finite-difference calculation procedure. The calculation procedure is based on the numerical algorithm called SIMPLE (for Semi Implicit Method for Pressure-Linked Equations) which was developed earlier by Patankar and Spalding [2] for parabolic flows. Because of the similarity in the equations for parabolic and partially-parabolic flows, the present calculation procedure shares many features with the parabolic one. In this paper importance is given to the differences between the two procedures; the common features are mentioned only briefly.

3.1. *Finite-difference equations*

The method of derivation of the finite-difference equations from the differential equations is identical to that in the parabolic calculation procedure [2]. The finite-difference equations are derived by integrating the differential equations over "control volumes" for individual variables transported. The three velocity components and pressure are stored in staggered positions on the finite-difference grid. The definitions of control volumes and storage of variables are shown in Fig. 1.

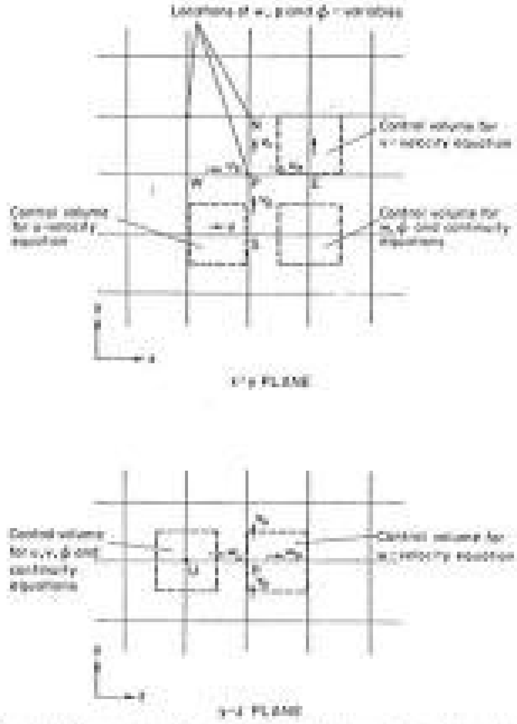

FIG. 1. Storage locations and control volumes for dependent variables.

The difference equations can be stated as follows:

Continuity:

$$C^u\{(\rho u)_e-(\rho u)_w\}+C^v\{(\rho v)_n-(\rho v)_s\}+C^w\{(\rho w)_p-(\rho w)_u\}=0 \quad (3.1)$$

Momenta:

$$u_P = A^u_N u_N + A^u_S u_S + A^u_E u_E + A^u_W u_W + B^u + D^u(p_P - p_W) \quad (3.2)$$

$$v_P = A^v_N v_N + A^v_S v_S + A^v_E v_E + A^v_W v_W + B^v + D^v(p_P - p_S) \quad (3.3)$$

$$w_P = A^w_N w_N + A^w_S w_S + A^w_E w_E + A^w_W w_W + B^w + D^w(p_U - p_P) \quad (3.4)$$

Property, ϕ:

$$\phi_P = A^\phi_N \phi_N + A^\phi_S \phi_S + A^\phi_E \phi_E + A^\phi_W \phi_W + B^\phi. \quad (3.5)$$

In the above equations, the A coefficients express the combined effects of convection and diffusion, linking the property at P with its neighbours in the cross-stream plane (Fig. 1); the B coefficients express the contribution of upstream convection and of source terms, expressed by "S" in the differential equations. The "C"s represent areas of cell faces across which mass is convected; and the "D"s are coefficients linking pressure differences to corresponding velocities. The subscripts P, N, S, E, W and U refer to variables at the grid nodes; and the subscripts p, n, s, e and w denote the variables at the interface locations shown in Fig. 1.

3.2. *Sequence of calculation steps*

The above difference equations are solved by an iterative procedure. All variables except the pressure are stored in two-dimensional arrays and are evaluated, over cross-stream planes, by marching in the predominant flow direction. The pressure field is stored three-dimensionally, and is first assigned a guessed value; it is then updated by sweeping repeatedly through the flow domain so as to remove errors in continuity and momentum.

The sequence of calculation steps is the following:

1. The three-dimensional pressure field is first assigned guessed values.

2. A march through the flow domain is initiated; and, from the inlet distributions of u, v and w their distributions at the next downstream location are calculated. The pressure gradient terms are evaluated from the guessed pressure field; and the coefficients A, B etc. are evaluated from variables in store at that instant. The equations are solved using a tridiagonal matrix algorithm (details are given in [2]).

3. The newly calculated distributions of u, v and w are checked for satisfaction of mass continuity at all the grid locations in the cross stream plane. The pressure and velocity fields are then corrected by solving a pressure-correction equation so as to remove

errors in mass continuity. The derivation and solution of the pressure-correction equation are described in the Appendix.

4. The equation for property ϕ is solved so as to provide distributions appropriate to the new downstream axial station.

5. Another new downstream axial station is chosen and the momentum, continuity and ϕ-equations are solved as described above. This step-wise march is continued until the end of the flow domain is reached. By the end of one complete marching sweep, a new three-dimensional distribution of pressure has been obtained.

6. Steps 2, 3, 4 and 5 are then repeated until the pressure corrections, or the continuity errors which give rise to them, have become smaller than a preassigned value. On the last sweep, the converged distribution of velocities, pressure, shear stresses, temperature etc. are printed out, as are needed.

3.3. *The boundary conditions*

The hydrodynamic boundary conditions governing the flow situation are prescribed through specified distributions of either velocities or pressure. When all the boundaries are of specified *velocity*, it is necessary, for *incompressible* flows, to fix one pressure point as a datum to the rest of the pressure field. In *compressible* partially-parabolic flows however, this is not necessary as the density level will decide the pressure level. The thermal boundary conditions are prescribed either as prescribed temperature or as prescribed heat flux at the boundaries.

4. AN APPLICATION OF THE CALCULATION PROCEDURE

This section describes an application of the calculation procedure. The physical situation considered is shown in Fig. 2: fluid flows through a square duct in which a wire screen is situated midway between inlet and outlet; the screen occupies only a portion of the cross-sectional area, and, in that region, creates a sink of axial momentum expressed by the relation

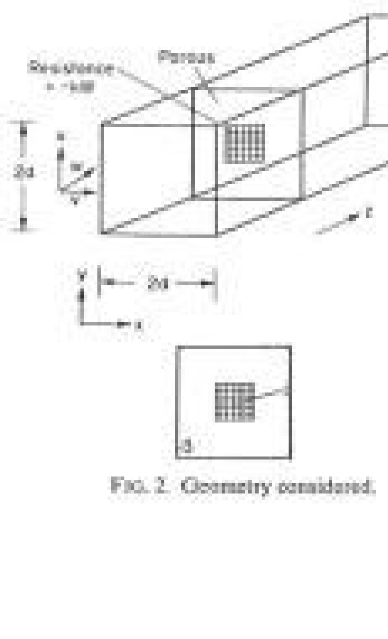

FIG. 2. Geometry considered.

$$S_w = -Kw \tag{4.1}$$

where K is a constant.

Because of the wire screen, the pressure in the centre of the duct rises to compensate for the additional pressure drop. This increase in pressure also retards the axial flow, thus diverting the streamlines away from the screen. The flow region further upstream of the screen also experiences the pressure rise and the bending of the streamlines, but to an extent diminishing with distance from the screen. Thus the flow is influenced by events downstream through the pressure field. The flow is partially-parabolic.

Calculations have been made for the above physical situation using the partially-parabolic calculation procedure. For ease of interpretation of the results, the duct walls have been considered to be frictionless, and the flow to be laminar. The finite-difference grid, for the typical calculations presented here, possessed 10 nodes in the x and y directions, and 40 grid nodes in the z-direction; the procedure, under the above conditions, converged in 18 sweeps of the flow domain.

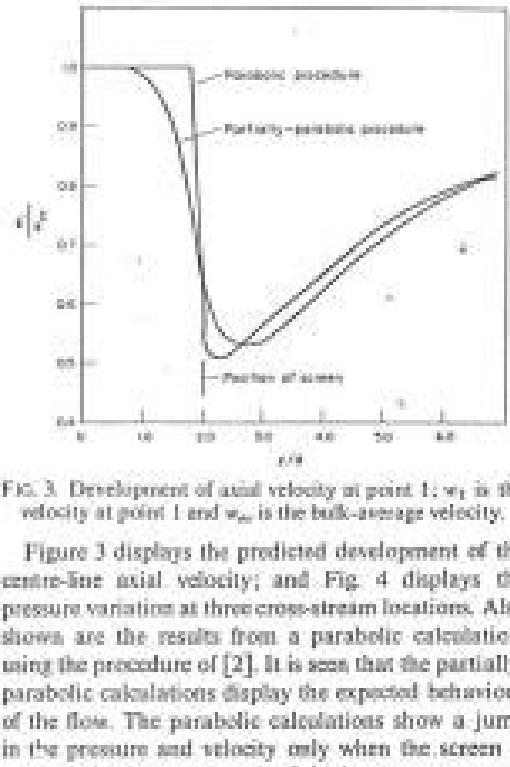

FIG. 3. Development of axial velocity at point 1; w_1 is the velocity at point 1 and w_{av} is the bulk-average velocity.

Figure 3 displays the predicted development of the centre-line axial velocity; and Fig. 4 displays the pressure variation at three cross-stream locations. Also shown are the results from a parabolic calculation, using the procedure of [2]. It is seen that the partially-parabolic calculations display the expected behaviour of the flow. The parabolic calculations show a jump in the pressure and velocity only when the screen is reached; further, as a result of the incorrect upstream flow field, the flow downstream of the screen is in error. It is therefore necessary to employ a partially-parabolic calculation scheme to predict the above flow situation.

5. CONCLUDING REMARKS

In the present paper, we have described a calculation procedure for partially-parabolic flow situations. Its benefits have been demonstrated by its application to a typical partially-parabolic flow problem.

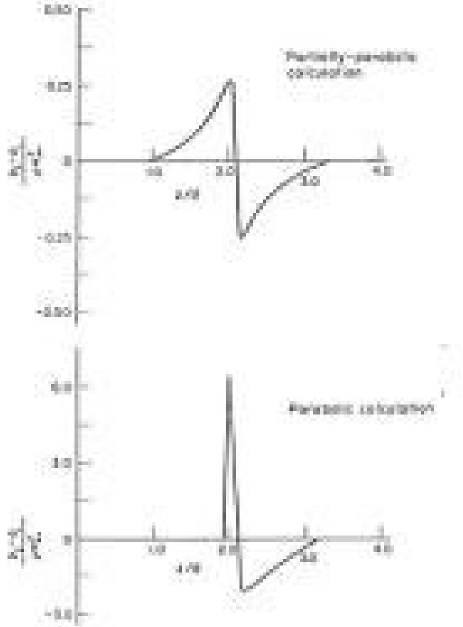

FIG. 4. Development of cross-stream pressure variation; points 1 and 3 are shown in Fig. 2; w_{av} is the bulk-average velocity. Note the difference in the vertical scales of the above two figures.

There are many practically occurring flow situations which need the present calculation procedure for their solution; a few of them are:

(a) Flow and heat transfer in a pipe bend, or in a tightly-wound spiral pipe.

(b) Flow of warm water from a power station, discharged into a river bend.

(c) Film cooling by discharge of coolant air from a row of inclined holes in the surface of a turbine blade.

(d) Flow in turbomachinery blade passages.

(e) The mixing of dilution air with combustion products in the downstream portion of a gas-turbine combustor.

Work on these is currently being conducted by the authors and their colleagues.

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REFERENCES

1. L. S. Caretto, A. D. Gosman, S. V. Patankar and D. B. Spalding, Two calculation procedures for steady three-dimensional flows with recirculation, in *Proceedings of Third International Conference on Numerical Methods in Fluid Dynamics*, Paris (July 1972).
2. S. V. Patankar and D. B. Spalding, A calculation procedure for three-dimensional parabolic flows, *Int. J. Heat Mass Transfer* **15**, 1787 (1972).
3. V. S. Pratap, Flow and heat transfer in curved ducts, Ph.D. Thesis, London University (1975).

APPENDIX

A1. *The Pressure-Correction Equation*

A1.1. *Derivation*

The purpose of the pressure-correction equation is to correct the pressure and velocity fields so that mass continuity is satisfied at all grid locations in the flow domain. The pressure-correction equation is derived from the continuity equation and simplified forms of the momentum equations. This appendix describes the derivation and solution procedure for the pressure-correction equation.

1. First, the velocity and pressure fields are expressed as:

$$p = p^* + p' \qquad u = u^* + u' \quad \text{and} \quad w = w^* + w' \qquad v = v^* + v' \qquad \text{(A1.1)}$$

where the primed quantities represent corrections to the best-estimate (asterisked) values.

2. The corrections to velocities are then related to the pressure corrections, by differentiation of the finite-difference momentum equations; only the central velocity of the control volume is allowed to vary during this differentiation. Thus:

$$u'_e = D^u(p'_P - p'_E) \qquad \text{(A1.2)}$$

where D^u is the coefficient in equation (3.2).

3. The expressions for velocity corrections, along with (A1.1), are substituted into the finite-difference form of the continuity equation; and the coefficients of pressure corrections are rearranged. The equation so derived is of the following form

$$A_P^p p'_P = A_N^p p'_N + A_S^p p'_S + A_E^p p'_E + A_W^p p'_W + A_D^p p'_D + A_U^p p'_U + m_P. \qquad \text{(A1.3)}$$

The A coefficients involve areas and the D^u, D^v and D^w coefficients. A_P^p is given by

$$A_P^p = A_N^p + A_S^p + A_E^p + A_W^p + A_D^p + A_U^p. \qquad \text{(A1.4)}$$

A1.2. *Solution of the pressure-correction equation*

Unlike the momentum equations, the pressure-correction equation has two additional terms $A_D^p p'_D$ and $A_U^p p'_U$ which link downstream and upstream pressure corrections to p'_P. Because of these links the equation is three-dimensional; i.e. a change of p' at D effects a change at U, and further upstream; so pressure-corrections need to be stored three-dimensionally. However, to avoid the three-dimensional storage of pressure-corrections, the pressure-correction equations are solved, in the present procedure, on cross-stream planes by repeated application of the tri-diagonal matrix algorithm. During the solution p'_U and p'_D are taken as equal to zero. By doing so, the pressures are updated as the marching sweep is contained.

In addition to the corrections dictated by the pressure-correction equation, the pressure field is further corrected in the following two ways. These two corrections have been observed to procure faster convergence of the numerical scheme.

1. A block adjustment (i.e. a uniform pressure increment over the plane) is applied at a plane downstream of that of P, to satisfy the overall mass-flow balance.

2. Certain fractions of the calculated pressure-corrections at any cross stream plane are applied also to pressures at upstream locations. The amounts of pressure-corrections depend upon the grid sizes, the nearness of the upstream location and the coefficients. It has been found that when this correction is made, the downstream events are transferred upstream at a faster rate. The details of the expressions used are given in [3].

ECOULEMENT TRIDIMENSIONNELS EN CONDUITE AVEC TRANSFERT THERMIQUE

Résumé—On décrit une procédure numérique de calcul applicable à des configurations tridimensionnelles d'écoulements en conduite de nature semi-parabolique, c'est à dire à des écoulements dans lesquels l'effet de la convection est dirigé longitudinalement vers l'aval, celui de la diffusion transversalement à l'écoulement, tandis que l'influence des régions en aval sur les régions en amont est transmise par le champ de pression. La procédure numérique permet de traiter économiquement de tels écoulements; la pression est mise en mémoire dans un tableau à trois dimensions tandis que les autres variables sont placées dans des tableaux à deux dimensions. A titre d'illustration, les résultats numériques obtenus dans une application de la procédure de calcul sont comparés à ceux obtenus à l'aide d'une procédure parabolique.

STRÖMUNGSVERHALTEN UND WÄRMEÜBERGANG IN DREIDIMENSIONALEN KANALSTRÖMUNGEN

Zusammenfassung—Es wird ein Berechnungsverfahren für dreidimensionale Kanalströmungen beschrieben. Es handelt sich hierbei um Verhältnisse teilweise parabolischer Natur, wobei konvektive Einflüsse sich nur stromabwärts auswirken, die Diffusion quer zur Strömung wirkt und lediglich über den Druck auch eine Beeinflussung entgegen der Strömungsrichtung auftreten kann. Die numerische Berechnungsmethode erweist sich als sehr geeignet für solche Strömungen; der Druck wird in dreidimensionaler Form eingegeben, während die anderen Variablen zweidimensional betrachtet werden. Zur Erläuterung werden für einen Anwendungsfall die Ergebnisse nach dieser Methode mit denen nach der parabolischen Berechnungsmethode verglichen.

ТЕЧЕНИЕ ЖИДКОСТИ И ТЕПЛООБМЕН В ТРЕХМЕРНОМ КАНАЛЕ

Аннотация — Рассматривается метод расчета трехмерных течений с профилем скорости, близким к параболическому, когда конвективный перенос имеет место только вниз по потоку, а диффузионный — поперек потока. Перенос из области вниз по потоку в область вверх по потоку осуществляется за счет давления. Численный расчет таких течений довольно прост. В этом случае давление является трехмерной величиной, а остальные переменные — двумерными. Для иллюстрации проведено сравнение полученных результатов с результатами расчета параболического случая.

Text to be
completed later

DRAFT

HTE ENGINEERING COURSE 1979/80 October 8, 1979

TEXT FOR REPORT REF. HTS/76/6 - "Basic Equations of Fluid Mechanics and Heat and Mass, and Procedures for their Solution".

Lecture 1

Panel 1

The course of lectures, of which this is the first, is structured in the manner indicated on the above panel.

Five lectures, included this introductory one, are devoted to fundamentals. In the present lecture the motivation of the course is provided; the following four lectures provide the basic laws of physics which form the starting point of the analysis, under the heading: conservation laws; flux laws; and source laws. These laws are put together, in the fifth lecture, to provide the fundamental differential equations to which solutions and solution procedures are to be sought.

It is rarely necessary, or indeed practicable, to provide solutions for the fundamental equations with all terms operative. Instead, "idealizations" are introduced which, by eliminating some of the complexities of real phenomena, reduce the difficulty and expense of the task of solution. It is usually intended that the eliminated physical phenomena are not of great significance for the purpose which the analyst has in mind; but sometimes the necessity to employ an inexpensive computational procedure entails more drastic discrepancies between the idealization and reality than can be comfortably tolerated. Much of the art of computational fluid mechanics is concerned with how to find the best idealization for particular processes, "best" implying a compromise between the requirements of realism and economy. Five lectures are devoted to idealizations of a general kind.

One particular kind of idealization is given a five-lecture sequence of its own. This is the breaking-up of the space-time continuum into a finite number of regions, variously called "control volumes", "cells", "elements", etc. Whereas the time dimension is ordinarily divided into intervals having the same magnitude for each part of the spatial domain of interest, the spatial dimensions themselves can be subdivided in a large number of different ways. These are described in the lectures; and their relative advantages and disadvantages are discussed.

The next sequence of lectures is concerned with classifying the problems which are commonly presented to the practitioner of computational fluid mechanics, and to the classification of the procedures which are available for solving those problems. The subject of classification is an important one, for, when properly understood, it enables the practitioner to match the solution procedure to the problem, avoiding the use of those procedures which would be either unsatisfactory in respect of accuracy, or excessively expensive for the task in hand.

The final seven lectures are devoted to illustrating some of these problems and procedures.

LECTURE 2

The objectives of the course are twofold; the first is concerned with understanding, and the second with technique.

The processes of heat, mass and momentum transfer in engineering and in the natural environment are, in their essence, both few and simple; but they can be combined together in so many ways as to produce an endless variety of processes. The lecture course will have been successful, in respect of the first of its objectives, if it permits the hearer or reader of the lectures to perceive the unity that pervades the diversity, and to be confident about his ability to distinguish the essential from the accidental.

However, the subject to be discussed is one which has great practical utility; for it permits, in a large number of cases, the practically significant features of natural and man-made fluid-dynamic phenomena to be predicted quantitatively.

This means, for example, that the probable performance of a piece of engineering equipment can be foretold, and assessed, without the necessity to construct or operate that equipment; and, particularly in the natural environment where experiments are difficult or impossible, predictions can be made which allow appropriate actions to be planned.

Often, especially in engineering, the ability to make quantitative predictions permits optimum designs of plant to be arrived at, and their optimum operating conditions to be determined.

Panel 3

The diagram on this panel is intended to portray the way in which the practice of quantitative prediction relates to engineering practice on the one hand and to research ("proving") on the other.

Were no costs involved in the translation of a design concept into immediate reality, and in the testing of the performance of that reality, it is the left hand-column of the panel which would occupy most attention in engineering schools and in the development divisions of industrial firms. The equipment has to provide a specified performance, as a rule; and whether it does so or not can certainly, in principle, be determined by fabricating the conceived design, and by testing the result. However, much engineering equipment is far too expensive for this to be a practical means of arriving at the design.

It is for this reason that quantitative prediction procedures are developed and used. The contents of such procedures are indicated in the middle column of the above panel, wherein it is seen that they rest on the laws of physics coupled with "models" of basic processes (to be discussed later). Mathematical principles are involved; the properties of materials and the special conditions of the equipment or process are also implicated; and this whole body of information is converted by a "prediction procedure" into the desired prediction of performance. Thus, the predictive column provides an alternative path from "design" to "performance".

Prediction procedures are often imperfect; and, even when they are very good, it is necessary to prove that they are reliable. Therefore, a large amount of research is concerned, nowadays, with testing the validity of prediction procedures. The contents of the "proving" box indicate what is involved: a special apparatus is devised, and operated upon by special measuring techniques; the measurements are processed and thereby turned into "results" which can be compared with predictions, made by a procedure ultimately intended for engineering use, of what the results of the measurements "should" have been. If the predictions and the results agree, the procedure is, to that extent, "validated"; and it can be used with somewhat more confidence by the engineering designer.

Panel 4

There now begins a list of processes involving heat, mass and momentum transfer to which it is appropriate, in practice, to apply prediction procedures of the kind which are discussed in these lectures. First, processes arising in the natural environment are listed.

The pollution of the ground-level atmosphere by smoke from chimneys, and the production of fog and even rain from the moist air which flows from the cooling towers associated with power stations, are nuisances with which industrial societies are all too familiar. The designers of the plants, if they could, would indeed design so as to minimise the nuisance; but it is not always easy to know, before the plant is built and its interaction with the terrain has been observed, how serious the nuisance will be. Both plant designers and environmental protection enthusiasts are therefore interested in being able to predict such processes quantitatively.

The prediction processes, which will be discussed in the following lectures, do permit the concentration distributions near industrial plant to be predicted; and they do so partly by way of a detailed prediction of the flow field. Such predictions, incidentally, can also furnish information about the forces exerted by the natural wind on structures.

There are similar processes, and prediction needs, connected with the hydrosphere. Warm water from power stations is poured into rivers and estuaries; and it is desired to know how far-reaching are its effects. Heat is lost from the surface of the water to the air; but damage may have been done to plant and animal life before sufficient heat has been rejected, and, if cooling water for the power plant is being withdrawn from the same body of water, there may be a recirculation of warm water which diminishes
What is true of thermal pollution is also true, with some modifications, of concentration pollution of natural waters.

It is often undesirable; and, when it cannot be prevented, one at least wishes to know its quantitative extent.

Heating and air-conditioning specialists are concerned with providing dwellings, concert halls, factories, etc. with air of the appropriate freshness and temperature for the comforts of the occupants. This air may be blown into a concert hall in sufficient quantity; but how can one tell how it distributes itself? A prodiction procedure which would indicate which seats were likely to be subjected to an unpleasant draught, and which were enveloped in stagnant air, would enable the designers to make changes before they were called upon to do so by called upon to do so by displeased concert-goers.

All the public buildings have, as far as possible, to be designed so that, should an accidental fire be present, the smoke which it produces should neither confuse nor asphyxiate the escaping human beings. For this reason, much attention is currently being given to the prediction of smoke movement in buildings by numerical means. For obvious reasons, it is not desirable to have to burn a building down in order to discover where the smoke is likely to go.

Beneath the ground also, there are processes of fluid and heat flow which are of vital importance. The production of oil from underground "reservoirs", the underground gasification of coal, the raising to the surface of salts by passing warm water through artificially-made underground cavities, and the use of the high temperature of some natural rocks for the production of steam and consequently of electric power, are all examples of which it is needless to emphasise the importance.

Panel 5

The needs of the aerospace industry have provided a major impetus to the development of prediction procedures. Although turbo-jet engines and rocket motors were at first developed before numerical computation was practicable, digital computers not having been invented, the expense of cut-and-try development proved to be so great that, as soon as computers and computational procedures were available, they were extensively used.

The above panel indicates several of the topics which have attracted particular attention. As will be evident to those who know something of the equipment in question, the topics range from those which are concerned solely with aerodynamics (for example the flow around a subsonic aircraft), through those which involve both fluid-dynamic and thermal effects (for example the cooling of turbine blades), to those in which chemical reaction and radiation also play significant parts.

The way in which such processes can be predicted will be described later in this lecture series.

Panel 6

The designer of a ship, and of the propellor which drives it, also has a need for computational fluid mechanics.

In the past, the performance of a ship in respect of frictional and wave-making drag was estimated from the measurements made on small-scale models towed along a long tank. This technique has never been entirely satisfactory, because the laws of dynamical similarity have prevented the maintenance of the equality of both the Froud and Reynolds numbers; and the prediction of the flow in the vicinity of the steering gear and propellors has been particularly inadequate.

Nowadays, therefore, much attention is being given to the numerical prediction of the flow around ships, with full account being taken of turbulence, free-surface effects, and the interactions between the rotating propellor and the nearby hull.

Such predictions are important not only from the point of view of designing for high-propulsive efficiency, but also, in the case of submarines which must be silent if they are to escape detection, for the prediction and hence the reduction of propellor-generated noise.

Panel 7

Fuel is in short supply, and is likely to become only more expensive as time goes on. Therefore, there is a great need to ensure that all equipment using fuel is as efficient as it can be.

Man has used fire for his purposes for perhaps one million years; and even modern fuel-using equipment has nearly all been designed on the basis of tradition modified by the impact of new materials and new needs. Furnaces, for example, have been designed hitherto, in just such "evolutionary" ways.

The same is true of internal-combustion engines of various kinds.

Now, however, strenuous efforts are being made to develop reliable numerical prediction procedures for the fluid-flow, heat-transfer and chemical-reaction processes which occur in furnaces and in the combustion chambers of engines. Although the difficulties are severe, they are being overcome.

Prediction procedures which are valid for wanted combustion processes are just as valid for those which are undesired. Part of the stimulus towards prediction procedures for combustion processes has therefore come from the side of fire-prevention and fire-control.

Panel 8

The above panel indicates some of the items of equipment which arise in the chemical-engineering and process industries. About each topic it would be possible to present one or more lectures. Here it seems best to advise the reader to explore his memory, library or such other sources of information as are available to him, so as to seek out a few particular examples of industrially-relevant embodiments of heat and mass transfer, and then to answer for himself the questions:

(a) How do the processes of fluid-flow, heat transfer, mass transfer and chemical reaction manifest themselves in this process?

(b) What are the costs associated with this process?

(c) What are the benefits associated with the benefits of the process?

(d) How could an ability to predict quantitatively the processes of fluid flow, heat transfer, mass transfer and chemical transformation, assist a designer to maximise the benefits and minimise the costs?

Panel 9

At a modest level, the prediction of the flow of heat and air through buildings has been practised for many years by heating engineers, who are required to determine how large a "radiator" to install, and air-conditioning experts have been concerned with how to preserve the humidity and temperature of the air within prescribed limits.

Nowadays, new demands are being made of predictive capabilities. More use is being made of the trapping of incident sunlight; in some areas of the world an increasing proportion of buildings are being placed underground, so that heat conduction through soil and rock becomes of especial importance; heat pumps, with their ancillary heat-exchanging devices, are receiving intensive study as fuel-saving devices; and, as fresh-air induction is reduced in order to diminish fuel consumption, it becomes more and more important to arrange the distribution system so that the air is optimally utilized.

It can not be said that, at the present time, mathematical modelling has played much part in the kitchen; nevertheless it is worth noting that cooking is, for the most part, a matter of carefully controlled mass transfer (i.e. mixing) heat transfer and chemical reaction. One day, perhaps, the computer will tell the cook how to bake a better cake!

Panel 10

There is much scope for the mathematical modelling of fluid-flow and heat and mass transfer processes in medicine. The circulation of the blood conveys nutrients and waste products to and from the various organs; and, through the enlargement or contraction of capillaries near the skin, the rate of heat loss of the body is controlled.

To the latter control device there is added the vapourization of moisture from skin, which is to be regarded as a combined heat-and mass-transfer process.

In part, the recognition of the role of heat and mass transfer in bodily functions may be regarded as just "interesting"; however, quantitative use can be made of predictive ability when any question of re-design arises. Thus, to introduce a "by-pass" into the circulation system is to modify, with beneficent intent, the pressure-drop~flow relationship of the appropriate part of the circulation system. This should be done on the basis of calculation rather than guesswork; and, further, it is desirable that some knowledge of the flow patterns induced at junctions should be available to the surgeon. The reason is that the blood cells are extremely sensitive to the shear stress in the stream; and, if excessive stresses are created, a break-down may occur.

Mathematical modelling of the flow of biological fluid is even more appropriate if these fluids are flowing outside the human body, for example in artificial lungs. There it becomes possible to design with greater precision than when organic tissue is in question. Although it cannot be said that human and animal biology forms a very large part of the field of applied mathematical modelling, it has its place, and it may grow.

Panel 11

Probably more mathematical-modelling activity in fluid mechanics and heat transfer is connected with nuclear power than with any other branch of industry. The reasons are not hard to see:-

(a) The capital costs of nuclear plant are very great.
(b) Once a plant has been started, it is extremely difficult to make subsequent changes to it because of the radioactivity which is engendered.
(c) Even under design conditions, the proper working of the power plant depends upon a careful balance between heat generation and heat loss in the reactor and elsewhere.

(d) The possibilities of departures from design conditions, which may be extreme, can be dangerous. It has therefore been necessary for the regulatory agencies to require safety devices which will deal satisfactorily with a number of possible, even although but remotely possible, accidents. For the most part, the question is: How can the coolant be brought to the over-heated equipment item before it is too late?

Problems of this kind occur whatever the nature of the reactor coolant. Possibly the most difficult mathematical-modelling problems occur when water is the coolant; for this boils and gives rise to a range of two-phase effects which have only recently come within the scope of mathematical modelling.

The fast-breeder reactor is cooled by liquid sodium which may also boil under accident conditions. The high conductivity of the sodium, as well as being beneficial from the point of view of coolant effectiveness, also creates certain problems for the mathematical modeller; it is no longer possible for him to separate so completely as he can from other fluids the laminar from the turbulent regimes.

Panel 12

Having completed a brief review of some of the more important physical processes which is the desire of the computational-fluid-mechanics expert to predict, a review, which, of course, could have been prolonged almost indefinitely, it is now appropriate to make some general remarks about the procedures of prediction which it is desired to produce.

It is obvious, although frequently given inadequate attention, that the ability to make predictions is worse than useless if the predictions themselves are wrong. The development of a prediction procedure is easy; but to show that its predictions are in accordance with reality is a task which can, strictly speaking, never be completed. What can be done, and should be done, is to test every prediction procedure against a large number of experimental situations of the kind for which the prediction procedure is likely to be used. If, in a high proportion of these cases, the predictions and the experimental findings agree, the procedure can be used with some confidence. Caution is, however, always necessary; and the less well "validated" the prediction procedure is, the greater the amount of caution that is necessary.

The second requirement of a useful prediction procedure is that it should be possible to ask it a question and obtain the answer in a short time. The total time expended has several components; and, often, the longest is that of learning how to use the procedure. Other components are mentioned on the panel.

Computer time costs money; and so does the time of those who must supply the information to the computer program, and interpret its output in the appropriate terms. Mathematical models would be used much more frequently if they were cheaper to use, that is to say if the computer expenses are not so great, and if all computer programs were so well automated that an engineer could address them without having to learn their specific language.

Two further requirements are mentioned on the panel, namely accessibility and flexibility. The former refers to the ease with which it is possible for an engineer with a definite problem to make connections with a computer which has the appropriate code mounted, and which will give him the answers he requires; the second relates to the ease with which a single computer code can respond to a large number of different questions. Both accessibility and flexibility are highly desirable.

Panel 13

Although computer codes have been mentioned above, and quite properly, it is worthwhile noting that "prediction procedures" are of many different kinds. A list is given on the above panel, with the simplest procedure at the top and the most complex at the bottom. As will be seen, it is the last-but-one that is of major concern of the present course of lectures.

The simplest prediction procedure is to suppose that that which worked last time will work next time if no changes are made. The only theoretical principle involved is what might be called "the uniformity of nature". It is a principle which, however, is not applicable when any novelty is present; and we need not consider it further.

The next kind of prediction procedure to be mentioned on the panel is that which postulates that components connected into a system will behave just as they do in isolation when the appropriate conditions are provided for them. This is a prediction procedure which is commonly used in plant design. For example, the compressor, combustion chamber, and turbine of a gas-turbine plant can be tested separately in the laboratory; then it can be supposed with good reason, that, when they are coupled together, they will behave in the same way. This is, indeed, how gas turbines, and many other items of equipment are designed.

Although "model" has been referred to earlier in this lecture, and will be referred to many times later, the third item in the panel refers to "<u>scale</u>" models, which are different. Here the reference is to the practice of making a dimensionaly correct simulacrum of an equipment item of interest, and operating it in a way which, because of the laws of dimensional similarity, will be quantitatively indicative of the way in which the full-scale equipment will perform. For example, a small-scale model of an aeroplane may be placed in a compressed-air tunnel; then the measurements which are made on it can quite accurately predict the lift and drag of the full-scale aircraft.

The scope for the performance of scale-model test is rather limited, because the laws of dimensional similarity often conflict. For example, it is known that, when a scale-model ship is to be towed in a tank, conditions may be found which represent the wave drag correctly but not the friction drag; or vice versa. In these circumstances, what on the panel are called "hopeful approximations" have to be made.

Ideally, predictions would also be made by the exact numerical solution of the fundamental differential equations governing the processes, without the introduction of any simplification. This, however, is not possible, in the majority of flow situations; usually the presence of turbulence is a sufficient obstacle to this path. The last entry in the panel is therefore one which is rarely put into practice.

The last-but-one, however, is the main subject of the present work; and it is characteristic of computational fluid mechanics as a whole. The differential equations are solved (or rather their finite-difference counterparts); but some of these equations are known to be at best approximate representations of reality. The task of research is to develop such "mathematical models" as to provide a good compromise between the penalties of excessive simplicity and excessive expense. This topic will be discussed in later lectures.

Panel 14

The central column of panel three is represented here in rather greater detail, with a somewhat different shape. "Prediction" appears in the heavy box in the top left hand corner; but it is here indicated that predictions are often used in order to provide optimal designs of equipment or processes.

At the bottom of the panel, the two general physical imputs, the "laws" and "models" are indicated. The "conservation", "flux" and "source" laws will be the subject of the next three lectures; and models of "turbulence", "radiation" and chemical "kinetics" will follow later.

The lecture course, will indeed consist largely of a progress up through the diagram. Differential equations will be described, and then their finite-difference counter-parts. These are inputs to computer programs.

On the right of the diagram it is seen that the "special conditions" which are appropriate to particular predictions often comprise information about geome y, about material properties, and about the operating circumstances of apparatus. These, together with the boundary conditions, complete the formulation of a mathematical problem. Thereafter it is simply necessary to have a computer code which operates on correct mathematical principles then a prediction procedure is in existence.

Panel 15

The final panel of this introductory lecture summarizes the present situation. A summary of the summary is: "doing well, but improvement needed".

From the practical point of view, mathematical modelling of fluid-flow, heat-transfer, mass transfer and chemical-kinetics processes is in full swing. Useful predictions are being produced, throughout the world, and in many branches of industry. Nearly always, the costs of doing the work are more than those who do it like to pay; and the reliability of the results still leaves much to be desired. However, there are few human activities of which similar remarks cannot be made.

Fortunately, it is known how, in all respects, improvements can be made; more research, more ingenious computational procedures, and a better business system for making the computational resources available for those who wish to use them. Progress can be expected to be rapid, and continuous over many decades.

DRAFT

HTE ENGINEERING COURSE 1979/80

LECTURE 2

Panel 1

There are four major conservation laws which underlie the theory of fluid mechanics and heat and mass transfer: they are the laws of conservation of: mass, momentum, chemical-species and energy.

Strictly speaking it is only mass which is truly conserved; therefore the word conservation has quotation marks around it in the second, third and fourth lines of the panel. What is meant by "conservation" is that it is possible to strike a balance between the factors effecting the change in the entity in question.

The conservation laws are among the most reliable in the annals of physics. Certainly they are more so than the "flux laws", to be dealt with in the next lecture, and the "source laws" to be dealt with after that.

The purpose of the present lecture is to represent the conservation laws in a compact mathematical form. It would be convenient if a coordinate-free notation could be adopted throughout, i.e. vector notation. Unfortunately, it is not easy to express all the terms in the differential equations in vector form; therefore the Cartesian-tensor form is employed in addition. However, the reader may be assured that no needless presentation of alternative but essentially equivalent forms of the equations will be made.

Panel 2

The law of conservation of mass, often called the "continuity equation", states that mass is neither created nor destroyed. This means that, if the amount of mass in unit volume changes, it has changed only as a result of, and in proportion to, the difference between the inflow and the outflow.

A vector-notation form is convenient. Here the operater "div" is a symbol of the net outflow of whatever appears in the bracket behind it. A more mathematical definition is given on the panel.

Panel 3

It may be as well to look first at the bottom line of the above panel, where the continuity equation is written in terms of Cartesian coordinates, whereby x_1 x_2 and x_3 are the three coordinate directions, and u_1, u_2 and u_3 are the velocity components in those directions. Most readers are probably familiar with the mass-conservation equation in this form.

At the top of the slide panel there is a boxed equation which represents the Cartesian-tensor equivalent of the equation just discussed. The subscript i stands for one of the three subscripts just mentioned; and the convention is such that if, in any term, subscript i appears twice, the term must be written out three times, with subscripts 1, 2 and 3 appearing in turn. Application of this rule to the equation at the top of the panel clearly will result in the equation at the bottom.

It should be emphasised that nothing is gained by the use of this convention apart from compactness. Any reader who tends to be confused by its use, and is willing to write out every equation at length, would be well-advised to do so.

Panel 4

There is a mathematical manipulation of the continuity equation which it will later be found convenient to use. The manipulation is to differentiate the density-velocity product by parts, and so, in ways which are hinted at on the panel, produce the boxed equation. This says that the divergence of the velocity vector (no longer the density-velocity-vector product) is equal to the negative of the "substantial derivative" of the logarithm of density. The importance of this formulation lies in the fact that it is often much easier to evaluate both the left-hand and the right-hand sides of the equation in this 'bulk-continuity' form than it is to evaluate the terms which arise in the form given in panel 3. Thus, the substantial derivative indicates the rate of change of the relevant property of a prescribed body of material. Often it will occur that, although sharp density variations exist within a flow, from point to point, or from time to time at a given location, the substantial derivative is still either zero or small. For two incompressible liquids, it is zero.

Readers who are more familiar with Cartesian coordinates might wish to satisfy themselves of the truth of the relation between the two forms of the continuity equation by performing a one-dimensional analysis.

<u>Panel 5</u>

Newton's second law of motion can be expressed in many ways. In the above panel, it is expressed in terms of a quantity, which can be thought of as a scalar, namely the amount of m-direction momentum per unit mass, u_m.

It will be recognised that momentum is a directional quantity: but, by concentrating on just one component of it, the m-direction component, it is possible to treat it as a scalar.

The boxed equation expresses, in vector notation, how the rate of change of the m-direction momentum per unit volume changes as a consequence of the transport of momentum across a control-volume boundary (the second term), and as a result of various forces (the right-hand side).

On that right-hand side there will be found terms expressing the stresses sustained by the fluid (div $\vec{p}_m$), the body force in the m-direction, for example gravity, and some postulated resistive force per unit volume, also in the m-direction.

The fluid-stress vector $\vec{p}_m$, can be regarded as the sum total of all the m-direction momentum transfers associated with molecular interaction within the fluid. Thus, both the direct stress (hydrostatic pressure), and the viscous momentum transfers, are comprehended within this term.

It is important to recognise that m-direction momentum is transferred in various directions, its own but also those at right angles to it.

The quantity f_m can be regarded as representing the force exerted on a fluid by a fixed solid porous medium through which it flows, as, for example, when oil flows through rock in a "reservoir" beneath the ground.

Panel 6

It is especially desirable to express the "momentum - conservation" equation in Cartesian-tensor form so as to facilitate, in the next lecture, the writing out in detail of the terms comprehended in $p_{m,i}$. The equation within the box in the above panel represents the first move in this direction.

It will be seen that, when the repeated-index convention is employed, the second term in the above equation is replaced by three terms, and so is the first term on the right-hand side.

It is worth noting that velocity appears in this equation, and in others, in two roles. First, it appears as it did in the continuity equation, in the role of a measure of the speed of change of position. This might be regarded as the "kinematic" role of velocity. Velocity appears also in the momentum equation in the role of momentum per unit mass; this may be regarded as its "dynamic" role. From many points of view, it would be desirable to have different symbols for the two concepts; and this is at least affected, above, by the use of differing subscripts.

Even when the direction i and the direction m are identical, it may be that u_i and u_m should be regarded as different. Thus, in a porous medium, which causes the fluid to accelerate between obstacles, it may be that the momentum per unit mass exceeds the average speed of forward motion.

Panel 7

It is common, and also quite useful, to effect a combination of the momentum and the mass conservation equations. This is achieved by differentiation by parts, and the results are expressed above. The details may be left to the reader who, if he is not familiar with vector algebra, will almost certainly be able to perform the equivalent transformation by working in Cartesian coordinates.

The form marked "alternative" above has a clear physical significance. It implies that the rate of change of the momentum per unit mass of a prescribed piece of material is influenced by the forces on the right-hand side of the equation.

Panel 8

Whereas panel 2 was concerned with the conservation of mass, regardless of its state of chemical aggregation, the next topic concerns the balance of processes effecting the mass of a given chemical species, denoted by subscript 1.

The mass of 1 per unit mass of mixture is given the symbol m_1, this implies that the mass of 1 per unit volume is $m_1\rho$. The question is: what makes the mass of 1 change with time?

According to the above statement of the "1-conservation" law, there are three influences, namely: convective flow, diffusive flow and chemical reaction.

The boxed equation expresses the same facts in mathematical form. The first term on the left-hand side represents the rate of change of the mass of 1 per unit volume; it is the "transient" term.

Next is seen the representation of the divergence of the sum of two vectors. The first vector, $m_1\rho\vec{u}$ represents the convected-mass-flow vector; it consists of the mass-flow vector regardless of species, $\rho\vec{u}$ multiplied by the mass of 1 per unit mass of mixture, m_1. The mass-flow vector has, of course, already appeared in the continuity equation and in the momentum equation.

Material is transfered across boundaries not only by "bodily transport", of the kind just referred to as convective mass flow; there is an additional mode, commonly called "diffusion", which is a consequence of relative motion of one species as compared with another in the mixture. In the above equation, the diffusion flux of 1 is given the symbol $\vec{J}_1$. In the next lecture, we shall have to consider how this diffusion flux is to be calculated.

The two terms operated upon by div represent flows across the boundary of a small control volume. By contrast, the term on the right-hand side of the equation refers to transactions which occur within the control volume. Specifically R_1 represents the mass rate of creation of chemical species 1 per unit volume. What it depends upon will be discussed in Lecture 4.

At the foot of the above panel three implications of the above definitions are noted. First, it is a consequence of the significance of m_1 that addition for all the species present in the mixture must equal unity; otherwise, all the components could not have been accounted for.

It is equally true that the sum of all the diffusion fluxes and of all the chemical reaction rates, over all species, must sum to zero. Thus, if the differential chemical-species "conservation" equations were written down for all the species in a mixture and added up, the result would be the equation on panel 2, as is to be expected.

Panel 9

What was done with the momentum equation can also be done with the chemical-species equation. First, the Cartesian tensor notation can be employed. Here the symbol $J_{l}i$, represents the i-direction component of the diffusive-flux vector.

Differentiation by parts, and combination with the continuity equation, permits the deduction of the second equation of the above panel. The substantial-derivative form, which shows that the mass fraction of l in a given body of mixture would remain constant were it not for diffusion and chemical reaction, is perhaps helpful to the understanding.

Other forms of the chemical-species "conservation" equation can be found in papers and text books. Often the mole fraction, of a partial pressure are prefered as indicators of the amount of a species which is present. Such formulations have their advantages; but they are difficult to use in the most general circumstances. The reason is that there is a law of mass conservation, but there is no law of mole conservation.

Panel 10

It is often useful to concentrate attention upon the mass of a chemical element per unit mass of mixture; for this may remain constant even though the mass fractions of a particular species are changing. In order to discuss this question, two new symbols are introduced above, namely m_{α} the mass of element α per unit mass of mixture, and $m_{\alpha,l}$ the mass of element α in unit mass of chemical species l.

As indicated above, m_{α} will ordinarily vary from place to place within a flow field; but $m_{\alpha,l}$ is a constant, depending only on the elemental composition of the species.

If the equation of panel 8 is multiplied by $m_{\alpha,l}$, and if then all such equations are summed for these species which contain the element α, the result is the above boxed equation. Examination of thes equation shows that it has a zero on the right-hand side, corresponding to the fact that there is no net source of the chemical element α. Summation signs have disappeared from the transient and the convective-mass-flow terms; but a summation sign remains for the diffusion terms. In a later lecture it will be seen how even this sign can disappear under some circumstances.

Panel 11

The first law of thermodynamics, or energy "conservation" law is harder to express in either words or symbols, because of the large number of forms of energy transport which in practice. Nevertheless a form similar to that of the equation on panel 8 may be discerned in the equation boxed above.

The transient term concerns the rate of increase of internal energy and kinetic energy per unit volume. This is here expressed by way of the stagnation enthalpy, which is the sum of the specific enthalpy and the kinetic energy; the stagnation enthalpy, $\tilde{h}$, with subtraction of the pressure divided divided by the density, represents the internal and kinetic energy per unit mass.

The first term operated upon by div represents the contribution of convective mass flow to the energy balance. It is the product of the mass-velocity vector and the stagnation enthalpy. Three other terms appear within the same bracket, they are :-

$\vec{Q}$, the heat-flux vector;
$\vec{W}_s$, the shear-work vector;
and the sum of the individual specific enthalpies of the species multiplied by their respected diffusion fluxes.

On the right-hand side is indicated the symbol S_{rad}, which stands for the radiative heat source. The three dots indicate that there can, in général, be other energy sources, for example, electrical heating.

Of course, before this equation can be used, it will be necessary to introduce expressions for all these terms.

Panel 12

The first two equations in the above panel represent simply the application to the energy equation of the same two manipulations as have already been seen, namely the introduction of the Cartesian-tensor form, followed by the combination with the continuity equation, which yields the equation for the substantial derivative of the stagnation enthalpy. That quantity, it is noted above, can also be expressed in terms of temperature, specific heat, and mass fractions. This extended form is more lengthy, but it is sometimes useful in leading to further simplifications.

Panel 13

Although it is premature to simplify the energy equation at the present point, the reader may be glad to receive the promise given above, of simplification to come later on.

There are many flows for which the shear-work and pressure-variation terms are negligible. This is true, for example, in a heat exchanger, in which the major temperature variations are the consequence of the heat transfer, and not of the flow work.

Then, the definition of a mixture specific heat, c, permits a simplification of the expression for the differential of the specific enthalpy, given above.

Further if it is permissible to regard the specific heats of all the species as being equal, as is often the case, a still further simplification arises. It is given above.

This matter will be returned to in lecture 7.

Panel 14

It is now appropriate to emphasise the similarity between the various equations by expressing them all in an identical form.

This is given in the boxed equation above, where the symbol ϕ stands for any of the variables which have been considered (u_m m_1 $\tilde{h}$) or for unity in the case of the continuity equation.

The substitution of unity for ϕ, coupled with the recognition that there is no net diffusion of mass, nor any net mass source, leads to the continuity equation of panel 2.

Substitution of m_1 for ϕ and of R_1 for S_ϕ leads to the recovery of the chemical-species "conservation" equation.

When $\tilde{h}$ or u_m are substituted for ϕ, the forcing of the "conservation" equations into the desired form can be effected only by making somewhat more arbitrary definitions of the diffusion flux and the source term. Just what is best to be done here will be discussed later, after the flux and source laws have been introduced.

Suffice it to say that from the point of view of numerical computation, it is extremely useful to be able to deal with all the major dependent variables in a common manner; for then thought is aided, computer programing simplified, and errors reduced. This theme will be elaborated throughout the lectures.

Panel 15

In conclusion it may be said that the equations which have been derived above have many counterparts throughout theoretical physics.

For example, the law of conservation of electrical charge can be expressed as above, although, in many cases, the mass-convective term is negligible.

Further, when the subject of turbulence models is discussed, it will be seen that the complexities of knowledge of turbulence flow can be usefully, and without too much departure from truth, condensed into equations having the same form as that of the general conservation equation on panel 14. This equation, incidentally, is sometimes called a "transport" equation. It is an equation which which we shall be largely concerned throughout this lecture course.

However, before attention can be given to the mathematical properties of the equation, it is necessary to establish how the term J_ϕ S_ϕ are to be evaluated. These topics are the subject of the next two lectures.